Tangent space symmetries: connections and coset decomposition

Tom Lawrence

December 23, 2017

Abstract

This paper looks at connections on a pseudo-Riemannian manifold and the symmetries of its tangent spaces. In particular, it looks at a coset decomposition of the general linear group of Jacobian matrices, and the relationship between this, the Levi-Civita connection and the Weitzenböck connection. I am intending to use this study as the starting point for a longer article.

1 Introduction

General relativity demonstrates the importance of geometry to our understanding of fundamental physics. The basic variables in the gravitational sector of the theory are the independent components of the metric. In an N-dimensional spacetime, there are $\frac{N(N+1)}{2}$ of these. Another key quantity is the connection - general relativity uses the Levi-Civita connection, which is uniquely defined on a given spacetime for a given coordinate system. The theory has been extraordinarily successful in describing the action of gravity.

When the theory was developed, the only other fundamental interaction that was recognised by physics was electromagnetism. Einstein was keen to extend the theory to incorporate electromagnetism in a geometric way. He tried a number of different approaches to this[1].

One approach was "Fernparallelismus", often called "distant parallelism" or "teleparallelism"[2]. He noted that an N-bein field has N^2 independent components. The components of the metric can be written as functions of these, but then there are $\frac{N(N-1)}{2}$ degrees of freedom contained in the N-bein field which describe invariances of the metric[3]. His idea was that these additional degrees of freedom could be used in describing electromagnetism. In defining an N-bein field across the spacetime manifold, he needed to use a new type of connection, which he discovered had already been investigated by Cartan, Weitzenböck and others[2, 4].

While this approach was unsuccessful in its aim, research into teleparallelism and its application to gravity has continued. It is now known that a theory of gravity can be based on the principles of teleparallelism which reproduces the field equations of general relativity. This is known as the Teleparallel Equivalent of General Relativity (TEGR). An excellent summary of the current state of knowledge can be found in a review by Pereira[5]; additional detail on aspects of this can be found in another review by Maluf[6].

The N-bein field components are the elements of a matrix transformation which maps a chosen frame basis into a coordinate basis, as described below. These matrices form a general linear group. Furthermore, the invariances of the metric described by Einstein form an orthogonal group, which is a subgroup of the general linear group. The general linear group can be partitioned into cosets of the orthogonal group and this leads to a natural decomposition of the change of frame.

This paper looks at the relationship between these connections and these tangent space symmetries, utilising the decomposition of the general linear group.

Where we need to specify the coordinate system a set of tensor components is in, we will do so by putting it in brackets in a superscript or subscript. For example, the components of a vector $\mathbf{V}$ in a coordinate system u'^M will be written $V^M_{(u')}$.

2 The tangent space at a point

General relativity treats spacetime as a curved Riemannian manifold, that is, one which approximates to flat space at each point. This allows one to define a tangent space at each point, the elements of which are vectors. By taking outer products of the tangent spaces and their duals, one can define tensors of higher rank.

On a curved Riemannian manifold, one needs to use curvilinear coordinates to parametrise a finite region of it. For a given region of any given manifold, there are an infinite number of possible curvilinear coordinate systems that could be used. General relativity is constructed to be generally covariant, making it easy to transform between different coordinate systems. The components of a vector can be transformed from one coordinate system to another using the Jacobian matrix for the transformation. Each such matrix is an element of a group isomorphic to $GL(4,\mathbb{R})$ (with matrix multiplication as the group operation). On an N-dimensional spacetime, Jacobian matrices are elements of a group isomorphic to $GL(N,\mathbb{R})$.

Consider an arbitrary N-dimensional Riemannian spacetime manifold $\mathcal{M}$. A region of it Ω may be parametrised using a system of curvilinear coordinates u^I. The vectors tangent to the curves of increasing u^1, u^2, ... at a point A form a basis for the tangent space $T_A\mathcal{M}$, denoted $\mathbf{e}_M|_A$ - the "coordinate basis" for u^M. (The coordinate basis may be similarly defined at any other point in Ω. Where we are evaluating a quantity at a given point, we shall state explicitly which point it is evaluated at, as the greatest source of errors in carrying out this research has been due to confusing an expression giving the value of a quantity at a given point with an expression for the quantity as a function.) The value of a vector field at A may then be written as a linear sum of this coordinate basis, and indeed any other coordinate basis:

$$\mathbf{V}|_A \in T_A\mathcal{M} = V^M_{(u)}|_A \mathbf{e}_M|_A = V^N_{(u')}|_A \mathbf{e}'_N|_A \tag{1}$$

The infinitesimal displacement vector $du^M|_A \mathbf{e}_M|_A$ in these bases is used as a template to derive the fundamental transformation law

$$V^M_{(u)}|_A \mathbf{e}_M|_A = V^M_{(u)}|_A \left.\frac{\partial u'^N}{\partial u^M}\right|_A \mathbf{e}'_N|_A \tag{2}$$

We can see this as a transformation of either the basis or the components.

This being a Riemannian manifold, we can define a symmetric inner product for each tangent space:

$$(\mathbf{V}, \mathbf{W})_A = (\mathbf{W}, \mathbf{V})_A \in \mathbb{R} \tag{3}$$

The image of this map on the coordinate basis is the metric at A:

$$g_{MN}|_A = (\mathbf{e}_M, \mathbf{e}_N)_A \tag{4}$$

and the inner product acts linearly over the tangent space. We can use this to find the transformation of the metric under a change of coordinates.

We can always define a set of coordinates x^I for which the basis is pseudo-orthonormal at our chosen point (with respect to the inner product). We will call this "frame basis" $\hat{\mathbf{n}}_I$:

$$(\hat{\mathbf{n}}_I, \hat{\mathbf{n}}_J)_A = \eta_{IJ} \tag{5}$$

The Jacobian matrix for transforming between this basis and the coordinate basis is an element of a group J_A which is isomorphic to $GL(N, \mathbb{R})$. We will denote the transformation between the chosen frame basis and the chosen (unprimed) coordinate basis $j_0|_A$:

$$(j_0)_M{}^I|_A = \left.\frac{\partial x^I}{\partial u^M}\right|_A \in J_A : \hat{\mathbf{n}}_M \mapsto \mathbf{e}_M = (j_0)_M{}^I \hat{\mathbf{n}}_I \tag{6}$$

while j will be used for a generic change of basis - for example,

$$j \in J_A : \mathbf{e}_M \mapsto \mathbf{e}'_M = j_M{}^N \mathbf{e}_N \tag{7}$$

Note that in this formalism, V^M consequently transforms according to:

$$j : V^M_{(u)}|_A \mapsto V^M_{(u')}|_A = V^N_{(u)}|_A (j^{-1})_N{}^M \tag{8}$$

As mentioned above, j_0 can be decomposed using a pseudo-orthogonal subgroup. Let $\mathcal{M}$ have t timelike dimensions and s spacelike dimensions. Then the Minkowski metric (5) is invariant under spacetime rotations (including boosts) and spacetime inversions (such as reflections) and combinations of these, which make up a group I_A isomorphic to $O(t,s)$. J_A can be partitioned into cosets of the form $\lambda_0 I_A$, so we can always write

$$j_0|_A = \Lambda_0|_A i_0|_A \tag{9}$$

where $i_0 \in I_A$. If we then define

$$\hat{\mathbf{k}}_K|_A = (i_0)_K{}^I|_A\ \hat{\mathbf{n}}_I|_A \tag{10}$$

we find that

$$(\hat{\mathbf{k}}_K, \hat{\mathbf{k}}_L)_A = \eta_{KL} \tag{11}$$

and

$$\mathbf{e}_M|_A = (\Lambda_0)_M{}^K|_A\ \hat{\mathbf{k}}_K \tag{12}$$

and

$$g_{MN}|_A = (\Lambda_0)_M{}^K|_A\ (\Lambda_0)_N{}^L|_A\ \eta_{KL} \tag{13}$$

3 Connections and covariant derivatives along a curve

Having examined the tangent space at a given point A, we now want to look at comparing the tangent spaces at different points. To do this, we need to use a connection.

General relativity uses a particular connection, the Levi-Civita connection, or Christoffel symbol. This has the advantages of being symmetric and being uniquely defined - on a given manifold in a given coordinate system, its components are single-valued at each point. However, when considering frame bases as we are here, it makes more sense to introduce the concepts by starting with connections on a curve, which can be generalised either to the Levi-Civita connection and its associated spin connection, or to those of teleparallelism.

Consider a curve $c(\lambda)$ through Ω parametrised by the single variable λ. We take λ to be invariant under changes of coordinate. Pick two points on it A and B. We define any map between the tangent spaces $T_A\mathcal{M}$ and $T_B\mathcal{M}$ which preserves linearity and the inner product as a "parallel map". There are an infinite number of these.

Now choose frame bases at both points, $\hat{\mathbf{n}}_I|_A$ and $\hat{\mathbf{n}}_I|_B$. Denote the parallel map $\tilde{}$ for which the image of $\hat{\mathbf{n}}_I|_A$ is $\hat{\mathbf{n}}_I|_B$:

$$\tilde{}: T_A\mathcal{M} \quad \rightarrow \quad T_B\mathcal{M} \tag{14}$$

$$\tilde{}: \hat{\mathbf{n}}_I|_A \quad \mapsto \quad \hat{\mathbf{n}}_I|_B \tag{15}$$

Then as $\tilde{}$ is a linear map,

$$\tilde{}: \mathbf{e}_M|_A \mapsto \tilde{\mathbf{e}}_M = (j_0|_A\ j_0{}^{-1}|_B)_M{}^N \mathbf{e}_N|_B \tag{16}$$

In the teleparallism formalism, this is valid regardless of how close or far apart A and B are. However, we are looking to define a connection. We therefore take A and B to be close to each other (in the language of relativity, the interval between these events is small). We then note that we can also define parallel maps to and from all the points on $c(\lambda)$ between these points - this set of parallel maps along this section of the curve constitutes a "parallelism". We choose this such that the transformation j_0 from the frame basis to the coordinate basis varies continuously with λ. (This means that not only must the coordinate basis and the frame basis be related by the same group J all along the curve, but j_0 must be in the same connected component of J at all points.) This allows us to carry out a Taylor expansion of ${j_0}^{-1}$ in λ, giving us

$$\tilde{\mathbf{e}}_M = \left(\mathbf{1} + \delta\lambda \left(j_0 \frac{\partial {j_0}^{-1}}{\partial \lambda}\right)_M{}^N\Big|_A\right) \mathbf{e}_N|_B + \mathcal{O}^2(\lambda) \tag{17}$$

From the linear nature of the parallel map, we then find the image of any vector $\mathbf{V}$:

$$\tilde{}: \mathbf{V}|_A \mapsto \tilde{\mathbf{V}} = V^N|_A \mathbf{e}_N|_B + \delta\lambda\, V^M|_A \left(j_0 \frac{\partial {j_0}^{-1}}{\partial \lambda}\right)_M{}^N\Big|_A \mathbf{e}_N|_B + \mathcal{O}^2(\lambda) \tag{18}$$

The quantity in brackets is our archetypal connection (up to a change in sign):

$$\left(\Gamma^{(u)}_\lambda\right)_M{}^N\Big|_A \equiv -\left(j_0 \frac{\partial j_0^{-1}}{\partial \lambda}\right)_M{}^N\Big|_A = \left(\frac{\partial j_0}{\partial \lambda} j_0^{-1}\right)_M{}^N\Big|_A \tag{19}$$

Under a change of curvilinear coordinates, from u^K to u'^K, we simply replace j_0 in these expressions by $j j_0$, where

$$j_M{}^N = \frac{\partial u^N}{\partial u'^M} \tag{20}$$

giving us

$$j: \left(\Gamma^{(u)}_\lambda\right)_M{}^N\Big|_A \mapsto \left(\Gamma^{(u')}_\lambda\right)_M{}^N\Big|_A = \left(j \Gamma_\lambda j^{-1}\right)_M{}^N\Big|_A - \left(j \frac{\partial j^{-1}}{\partial \lambda}\right)_M{}^N\Big|_A \tag{21}$$

One possible change of coordinates is to the set x^I mentioned above, with pseudo-orthonormal basis at A. Then $j = j_0^{-1}$, so

$$\left(\Gamma^{(x)}_\lambda\right)_M{}^N\Big|_A = 0 \tag{22}$$

If $c(\lambda)$ is a geodesic, then x can have pseudo-orthonormal basis, and $\Gamma^{(x)}_\lambda = 0$, along the entire curve.

We can also look at changing parallelism. Consider a new parallelism $\bar{}$, which again preserves orthonormality, so that

$$\bar{}: \hat{\mathbf{n}}_I\big|_A \mapsto \bar{\mathbf{n}}_I\big|_B = i_I{}^J\, \hat{\mathbf{n}}_J\big|_B \tag{23}$$

If i is constant along $c(\lambda)$, Γ_λ is unaffected. But if i varies with λ (we take it to be in the same connected component of I at every point),

$$\bar{} : \left(\Gamma_\lambda\right)_M{}^N\Big|_A \mapsto \left(\Gamma'_\lambda\right)_M{}^N\Big|_A - \left(j_0 i \frac{\partial i^{-1}}{\partial \lambda} j_0^{-1}\right)_M{}^N\Bigg|_A \tag{24}$$

We can use (18) to define a covariant derivative:

$$D_\lambda V^N = \partial_\lambda V^N + V^M \left(\Gamma_\lambda\right)_M{}^N \tag{25}$$

It is easy to show that this transforms covariantly:

$$j : D_\lambda^{(u)} V_{(u)}^M \mapsto D_\lambda^{(u')} V_{(u')}^M = D_\lambda^{(u)} V_{(u)}^N \left(j^{-1}\right)_N{}^M \tag{26}$$

In the x coordinates, this simply becomes

$$D_\lambda^{(x)} V_{(x)}^M = \partial_\lambda V_{(x)}^M \tag{27}$$

4 Connections and covariant derivatives across Ω

It is possible to extend the way we defined Γ above to the whole of Ω. Rather than just defining a parallelism - a set of parallel maps - along a curve, we define a parallelism across the whole of Ω. This results in j_0 becoming a field over u^I. We can then define a connection field using the same approach as in (17), except we now Taylor expand in each of the curvilinear coordinates; this is known as the Weitzenböck connection:

$$\dot{\Gamma}_{LN}{}^M(u) \equiv -\left(j_0 \partial_L j_0{}^{-1}\right)_N{}^M \equiv \left(\partial_L(j_0) j_0{}^{-1}\right)_N{}^M \tag{28}$$

This is not the most general connection. Other rules for parallel transporting a vector exist, which do not take this form. More generally,

$$\tilde{}: \mathbf{V}|_A \mapsto \tilde{\mathbf{V}} = V^N|_A \mathbf{e}_N|_B - \delta u^L\ V^M|_A \Gamma_{LM}{}^N \mathbf{e}_N|_B + \mathcal{O}(\delta u)^2 \tag{29}$$

The transformation of $\Gamma_{LM}{}^N$ under a local change of basis is similar to the transformation for Γ_λ, except that we now need to act on the index L:

$$j(u) : \Gamma_{LM}^{(u)\ N} \mapsto \Gamma_{LM}^{(u')N} = j_L{}^K \left(j \Gamma_K^{(u)} j^{-1}\right)_M{}^N - j_L{}^K \left(j \partial_K j^{-1}\right)_M{}^N \tag{30}$$

where $(\Gamma_L)_M{}^N \equiv \Gamma_{LM}{}^N$.

Just as for Γ_λ, we can apply a transformation j_0^{-1} to reduce the Weitzenböck connection to zero - except that we can now do it over the whole of Ω. However, on a curved manifold, the frame bases defined by

$$\hat{\mathbf{n}}_I = \left(j_0^{-1}\right)_I{}^M \mathbf{e}_M \tag{31}$$

at each point do not represent the basis for any coordinate system.

It is worth noting what happens on a geodesic in more detail. If we consider a point particle moving along a geodesic, we can always base a set of coordinates x^I on its rest frame. The geodesic is parametrised by τ, the particle's proper time, which is proportional to x^0:

$$x^0 = c\tau \tag{32}$$

These coordinates are "Riemann normal coordinates": they have pseudo-orthonormal basis along the entire geodesic, and indeed the first derivatives of the metric are zero. By comparison with (22), we therefore have

$$\left.\dot{\Gamma}^{(x)\ N}_{0M}\right|_{c(\lambda)} = 0 \tag{33}$$

For any connection $\Gamma^{(u)\ N}_{LM}$, we may define the covariant derivative of a vector, with components

$$D_L V^M = \partial_L V^M + V^N {\Gamma_{LN}}^N \tag{34}$$

The covariant derivative at a point A is an element of $T_A\mathcal{M} \otimes T^*_A\mathcal{M}$. Under a local change of basis, the inhomogeneous term in the transformation of Γ is cancelled by the inhomogeneous term in the transformation of $\partial_L V^M$. Consequently, $D_L V^M$ transforms covariantly:

$$j : D^{(u)}_L V^M_{(u)} \mapsto D^{(u')}_L V^M_{(u')} = {j_L}^K D^{(u)}_K V^N_{(u)} \left(j^{-1}\right)_N{}^M \tag{35}$$

This can be extended in the normal way to tensors of other ranks.

It is easy to show that any connection for which (29) preserves the inner product of vectors is metric compatible, that is

$$D_L g^{MN} = 0 \tag{36}$$

However, it is not necessarily symmetric. For example, the Weitzenböck connection is metric compatible, but has a torsion:

$$\dot{T}_{LM}{}^N = \dot{\Gamma}_{LM}{}^N - \dot{\Gamma}_{ML}{}^N \neq 0 \tag{37}$$

The only symmetric, metric-compatible connection is the Levi-Civita connection:

$$\mathring{\Gamma}_{LM}{}^N = \mathring{\Gamma}_{ML}{}^N = \frac{1}{2} g^{NK} \left(\partial_K g_{LM} - \partial_L g_{KM} - \partial_M g_{KL}\right) \tag{38}$$

Now for any geodesic $c(\lambda)$, in the Riemann normal coordinates x^I, the derivatives of the metric are zero, so

$$\left.\mathring{\Gamma}^{(x)\ N}_{LM}\right|_{c(\lambda)} = 0 \tag{39}$$

However, away from the geodesic the Levi-Civita connection is non-zero on a curved manifold, even in this coordinate system. Note that incorporating (33), we have

$$\left.\dot{\Gamma}^{(x)\ N}_{0M}\right|_{c(\lambda)} = \left.\mathring{\Gamma}^{(x)\ N}_{0M}\right|_{c(\lambda)} = 0 \tag{40}$$

We conclude this section by noting some further properties of the Weitzenböck and Levi-Civita connections. The Weitzenböck connection has zero field strength[5, 7]:

$$\partial_L \dot{\Gamma}_{NK}{}^M - \partial_N \dot{\Gamma}_{LK}{}^M + \dot{\Gamma}_{NK}{}^J \dot{\Gamma}_{LJ}{}^M - \dot{\Gamma}_{LK}{}^J \dot{\Gamma}_{NJ}{}^M = 0 \tag{41}$$

and (as noted above), it can be reduced to zero across Ω by a local change of basis. The scalar curvature (the Ricci scalar) may be constructed from its torsion tensor[5, 6]. For a given coordinate system on a given manifold, this connection is not unique - its definition depends on the parallelism chosen.

The field strength of the Levi-Civita connection is the Riemann curvature tensor:

$$R^M{}_{KLN} = \partial_L \mathring{\Gamma}_{NK}{}^M - \partial_N \mathring{\Gamma}_{LK}{}^M + \mathring{\Gamma}_{NK}{}^J \mathring{\Gamma}_{LJ}{}^M - \mathring{\Gamma}_{LK}{}^J \mathring{\Gamma}_{NJ}{}^M \tag{42}$$

and the connection cannot be reduced to zero across Ω by a local change of basis, except on a flat spacetime. For a given coordinate system on a given manifold, it is unique. The Riemann tensor can also be viewed in terms of the action of the covariant derivatives on a vector field:

$$[D_K, D_J]\, W^I = R^I{}_{LKJ} W^L \tag{43}$$

Finally, each connection $\Gamma_{LM}{}^N$ has an associated Lorentz connection or spin connection. Pereira[5] defines a Lorentz connection as a one-form assuming values in the Lie algebra of the Lorentz group. In N dimensions, this will be in the Lie algebra $SO(1, N-1)$. This means that at least two of its indices must be frame indices. It therefore has two forms, one of which has all three indices as frame indices, while the other has two frame indices and one coordinate index. In the formalism of this paper, the Lorentz connection with three frame indices is considered to be the usual connection in the frame basis. The frame basis at a point A is the basis at that point for some set of Riemann normal coordinates x^I, so we can write this connection at this point as $\Gamma^{(x)}_{LK}{}^M\Big|_A$. The form with two frame indices and one coordinate index is considered to be in a mix of two different bases. We shall write this as follows:

$$\omega_M{}^{IJ} T_{IJ} \in SO(1, N-1) \tag{44}$$

where the first index is taken to be a coordinate index and the last two are frame indices.

If we choose a frame $\hat{\mathbf{n}}_M$ at A related to the coordinate basis by (**??**) where j_0 can be decomposed using (**??**), any connection in the coordinate basis can be related to a Lorentz connection as follows:

$$\Gamma^{(u)}_{LM}{}^N = (\Lambda_0)_L{}^K \eta_{KI} \omega_M{}^{IJ} (\Lambda_0^{-1})_J{}^N + (\Lambda_0)_L{}^K \partial_M (\Lambda_0^{-1})_K{}^N \tag{45}$$

This equation can be inverted to give:

$$\omega_M{}^{IJ} = \eta^{IK} [(\Lambda_0^{-1})_K{}^L \Gamma^{(u)}_{ML}{}^N (\Lambda_0)_N{}^J + (\Lambda_0^{-1})_K{}^L \partial_M (\Lambda_0)_L{}^J] \tag{46}$$

Of course, this is not unique: any local change of frame $i(u)$ (including $i_0(u)$) results in another Lorentz connection. ${\omega_M}^{IJ}$ transforms under a local change of frame according to:

$$i(u) : {\omega_M}^{IJ} \mapsto {\omega'_M}^{IJ} = \left(i\omega_M i^{-1}\right)^{IJ} - \left(i\partial_M i^{-1}\right)^{IJ} \tag{47}$$

where frame indices are raised and lowered using η^{IK} and η_{IK}. It transforms under a change of curvilinear coordinates according to:

$$j(u) : {\omega_M}^{IJ} \mapsto {\omega'_M}^{IJ} = {j_M}^N {\omega_N}^{IJ} \tag{48}$$

Note that the Weitzenböck spin connection can be reduced to zero everywhere by a local change of frame, whereas the Levi-Civita spin connection cannot[5].

References

[1] Tilman Sauer 2007 *Einstein's Unified Field Theory Program*

[2] Tilman Sauer 2006 *Historia Mathematica* **33** 399-439

[3] Albert Einstein 1928 *Sitzber. Preuss. Akad. Wiss.* **17** 217-22; translated into English in arXiv:physics/0503046 [**physics.hist-ph**] and [4] below

[4] *Selected Papers on Teleparallelism, translated and edited by* D. H. Delphenich

[5] J G Pereira 2012 *AIP Conf.Proc.* **1483** 239-259 arXiv:1210.0379 [**gr-qc**]

[6] José W Maluf 2013 *Ann. Phys. (Berlin)* **525** No.5, 339-57

[7] A.P. Balachandran, A. Stern, C.G. Trahern, *Phys. Rev. D***19**(1979) 2416-2446