Non-linearly realised O(3) symmetries

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Abstract

Sigma models or non-linear realisations are often seen in three different ways: firstly, as the result of placing a constraint on a linear multiplet of fields, secondly as a field space which is diffeomorphic to a coset space, and thirdly as the low-energy limit when the global symmetry of a system of fields is spontaneously broken. In these notes we show the equivalence of these three approaches for the case where O(3) symmetry is broken and the constrained/low-energy field space is $S^2 \equiv \mathrm{SO}(3)/\mathrm{SO}(2)$. The notes are in two parts. The first part is a study of the differential geometry of S^2 field space and its diffeomorphism to SO(3)/ SO(2), while the second part deals with the spontaneous breaking of O(3) symmetry, both global and gauged, and covers related field theory issues such as the inclusion of fermions, supersymmetry and the quantisation and renormalisation of the models.

In the first part, a constraint is placed on a triplet of scalar fields to parametrise a two-sphere. We find that the fact that this model is so easy to visualise greatly clarifies the relationship between the field space, the connected component of the space of isometries SO(3) and the coset space SO(3)/ SO(2). We work with three well-known coordinate systems and find the variations in the coordinates and the transformations of vectors in the spaces tangent to the manifolds. The metric for the space and consequently the Lagrangian for the system are calculated in a number of ways. It is clear that the conceptual structure of this study may be readily applied to other non-linear realisations.

In the second part, two models are studied in which an internal O(3) symmetry is spontaneously broken. Both contain a triplet of scalar fields. In the first, these constitute the entire field content of the model and there is a global O(3) symmetry. The methods of Salam and Strathdee are used to express the Lagrangian in the language of non-linear realisations and it is explicitly demonstrated that in the low-energy limit this becomes the Lagrangian of the O(3) non-linear sigma model. It is then shown how this model relates to the Georgi-Glashow model, in which the triplet is coupled to an SO(3) gauge field. After applying the methods of Salam and Strathdee, the field content and properties of the gauged theory are as expected from Higgs-Kibble theory. These methods can easily be extended to other symmetry-breaking patterns and this study may help improve understanding of some of the geometric aspects of Seiberg-Witten theory.

Part I
The pure classical O(3) sigma model

1 Introduction to Part I

The concept of spontaneous symmetry breaking is a crucial part of the Standard Model and extensions of the Standard Model are largely based on spontaneously breaking higher-dimensional symmetry groups to those of the Standard Model. The low-energy effective theories are best described by non-linear realisations or sigma models, so it is perhaps not surprising that since these were first discovered in the 1960s[1]-[7] they have kept cropping up, for example, in supergravity[8]-[12], Seiberg-Witten theory[13, 14], the theory of D-branes[15, 16] and models of nucleons[17].

It could be argued that the greatest advances in the field of non-linear realisations have been made by researchers who were not considering particular symmetries but rather, general features of all non-linear realisations. These features were often obscured in early work by the focus on chiral symmetries.

What is presented in these notes is the simplest possible model, in which these concepts are demonstrated on a mathematical space familiar from everyday life - the surface of a sphere. This model has long been of interest for its topological properties - there is a wealth of research into classical field configurations and their quantum corrections, particularly in lower-dimensional spacetime. However, the fact that all the important features may be so readily visualised also makes it the perfect tool for those wishing to get to grips with the geometry of sigma models, non-linear realisations and spontaneous symmetry breaking[1]. These concepts can then readily be applied to other symmetry-breaking schemes.

The phrase 'sigma model' originates from a seminal paper by Gell-Mann and Levy[1], which describes a model resulting from constraining the norm of a multiplet of four Lorentz scalar fields. In the O(3) non-linear sigma model, precisely the same is done for a triplet of such fields. The resulting field space inherits the O(3) symmetry of the Euclidean field space, but is a curved space on which different coordinate systems are useful in different situations. Meetz[7] used coordinates which are the three-dimensional version of what we call 'projective coordinates' in these notes and was the first to develop the concept of a metric for the space. Coleman *et al*[5] noted that all coordinate systems representing physical fields must have their origin at the same point and identified a coordinate system which may be used for any field space which is isometric under G a compact, connected semisimple Lie group and whose vacuum point is

[1] In Section 11 of Part II, we use the methods of Salam and Strathdee[18] to show how this model arises naturally through spontaneous symmetry breaking.

invariant under H a proper Lie subgroup. (We call these 'standard coordinates'; our third coordinate system is the familiar spherical polar coordinates.)

In a series of papers, Isham[6, 19, 20] considered the relationship between the group manifold G, the coset space G/H, the curved field space and any flat space it may be embedded in. However, the complexities of chiral symmetry made it difficult to get to grips with the geometrical concepts. All of the above have been major influences on these notes. (More recently, Maison[21] and Coquereaux[22] have also looked at the geometry of sigma models in general.)

Interest in the O(3) non-linear model started in the mid 1970s as a result of similarities with models of ferromagnets[23, 24, 25], with the focus being on classical field configurations. Soon after this, it became clear that despite the isometry group G of a non-linear sigma model being a global one, non-linear sigma models admit a local H symmetry[26, 27] and can even be rewritten in a form which includes auxilliary gauge fields for H[28, 29, 30]. We will review in outline how this can be done for our model.

Quantum field theory textbooks (for example, Ryder[31]) often use the model considered here in their introduction to spontaneous symmetry breaking, because when O(3) symmetry is spontaneously broken, the spherical field space represents the vacuum manifold and its origin the chosen vacuum state. The arguments presented in the textbooks, however, are often fairly heuristic; the author is not aware of any work which treats this model with the rigour of these notes.

The article which is most closely related to Part I of these notes is that of Barnes, Generowicz and Grimshare[32]. They carry out the same algebraic analysis of non-linearly realised SU(2) (linear when restricted to a U(1) subgroup) as we do for SO(3) (linear when restricted to an SO(2) subgroup), although they omit the geometrical analysis. Given that the adjoint map provides a homomorphism from SU(2) to SO(3) (mapping the U(1) subgroup to the SO(2) subgroup), we would expect the quantities we calculate to take the same algebraic form as theirs and we shall see this is indeed the case. (SU(2) group elements provide a matrix representation of the quaternion algebra and it is well known that quaternion operations can provide a simpler way of carrying out rotation group calculations - see for example Kuipers[33]. Some recent papers make use of this[34, 35].) Most recently, calculations for this sigma model have been performed by Hamilton-Charlton[36] and we shall eventually see how our results are equivalent to his.

Part I is divided into different sections dealing with different manifolds: Section 2 uses $\mathbb{R}^3$ to set up some basic conventions, Section 3 is concerned with S^2, Section 4 is concerned with the SO(3) manifold and Section 5 with the coset space SO(3)/SO(2) and its diffeomorphic subspace in SO(3). The discrete $\mathbb{Z}_2$ symmetry is dealt with separately in Section 6 and the gauging of the H symmetry of the model in Section 7. This concludes a rigorous analysis of the geometry of the pure sigma model (with just the two independent scalar field components). Throughout this analysis, nothing is assumed about the dimensionality of the underlying spacetime. The final section of Part I contains a very brief summary of known solutions of the classical equations of motion of this model, in different numbers of spacetime dimensions.

Finally, we should note that while this is intended to be the most complete and rigorous study of this model yet, the author also wishes to make it accessible to those who have a strong background in physics rather than pure geometry. Therefore we do not follow the convention in modern geometry texts of taking basis vectors to be differential operators, instead we use the more abstract notation of $\hat{\mathbf{e}}_i$, which many physicists will be more familiar with. Where it is unclear, we give these a two-letter superscript in Roman type in brackets. The first denotes the manifold: R for $\mathbb{R}^3$, S for S^2, G for the SO(3) group manifold and C for the coset space. The second denotes the coordinate system: C for Cartesian, p for projective and s for standard.

Also, we do not explicitly use the concept of a dual vector space. Vectors are written as linear sums of basis vectors; the latter have covariant indices while the components have contravariant indices. An inner product is defined for each tangent space and the metric is the inner product of the basis vectors. For each contravariant quantity, a covariant counterpart can be defined using the metric. This should all become clear in Section 2.

2 The $\mathbb{R}^3$ manifold

2.1 Basic conventions

In Cartesian coordinates, a point may be labelled

$$\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z \tag{1}$$

where $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ are orthonormal basis vectors in the directions of increasing x, y, z:

$$(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j) = \delta_{ij} \; . \tag{2}$$

In spherical polars,

$$\mathbf{r} = r\hat{\mathbf{e}}_r(\theta, \phi) \tag{3}$$

where

$$\hat{\mathbf{e}}_r = \sin\theta\cos\phi\hat{\mathbf{e}}_x + \sin\theta\sin\phi\hat{\mathbf{e}}_y + \cos\theta\hat{\mathbf{e}}_z \; . \tag{4}$$

We can also define basis vectors in directions of increasing θ and ϕ, normalised so that

$$\mathrm{d}s^2 \equiv (\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{r}) \tag{5}$$

is the same in either coordinate system:

$$\hat{\mathbf{e}}_\theta = r\cos\theta\cos\phi\hat{\mathbf{e}}_x + r\cos\theta\sin\phi\hat{\mathbf{e}}_y - r\sin\theta\hat{\mathbf{e}}_z \; , \tag{6}$$

$$\hat{\mathbf{e}}_\phi = -r\sin\theta\sin\phi\hat{\mathbf{e}}_x + r\sin\theta\cos\phi\hat{\mathbf{e}}_y \; . \tag{7}$$

The corresponding coordinate transformations are well known. These basis vectors are orthogonal, so the metric is diagonal:

$$g_{ij} = (\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} . \tag{8}$$

The components of a vector $\mathbf{V} = V^i \hat{\mathbf{e}}_i$ transform under coordinate transformations according to the usual rule for contravariant vectors. To ensure that $\mathbf{V} = V^i \hat{\mathbf{e}}_i = V'^i \hat{\mathbf{e}}'_i$, the basis vectors must transform as covariant vectors. $\mathbf{r}$ transforms as a vector only in flat space but $\mathrm{d}\mathbf{r}$ transforms as a vector in any space.

2.2 Isometries and Killing vectors

Isometries are (diffeomorphic) active transformations which preserves the length of $\mathrm{d}\mathbf{r}$:

$$\begin{aligned} &(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{r}) = \mathrm{d}x^i \mathrm{d}x^j (\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j) = \mathrm{d}x^i \mathrm{d}x^j \; g_{ij}|_P \\ &= (\mathrm{d}\mathbf{r}', \mathrm{d}\mathbf{r}') = \mathrm{d}x'^i \mathrm{d}x'^j (\hat{\mathbf{e}}'_i, \hat{\mathbf{e}}'_j) = \mathrm{d}x'^i \mathrm{d}x'^j \; g_{ij}|_{P'} \end{aligned} \tag{9}$$

or

$$\frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} \, g_{ij}|_{P'} = g_{kl}|_P \; . \tag{10}$$

For $\mathbb{R}^3$, these are translations and orthogonal transformations. In this section and the next three, we will focus on continuous transformations from $\mathbf{r}$ to $\mathbf{r}'$, so we restrict the latter to the group SO(3), representing rotations about the origin. An arbitrary matrix of this group may be written

$$R = \mathrm{e}^{\mathrm{i}\omega^i T_i} = \mathbb{1} + \mathrm{i}\omega^i T_i + \mathcal{O}(\omega^i)^2 \tag{11}$$

where the generators T_i satisfy

$$[T_i, T_j] = \mathrm{i}\epsilon_{ij}{}^k T_k \; . \tag{12}$$

Any one-parameter subgroup acting on $\mathbf{r}$ traces out a curve; a tangent vector to this curve at $\mathbf{r}$ is a Killing vector. There is one independent Killing vector for each parameter ω^i, e.g.

$$\left.\frac{\mathrm{d}x'^i}{\mathrm{d}\omega^1}\right|_{\omega^1=0} \hat{\mathbf{e}}_i = \left.\frac{\mathrm{d}R^i{}_j(\omega^1)x^j}{\mathrm{d}\omega^1}\right|_{\omega^1=0} \hat{\mathbf{e}}_i = K^i{}_1 \hat{\mathbf{e}}_i \equiv \mathbf{K}_1 \; . \tag{13}$$

These are first-order variations in x^i in a Taylor series. The Killing vector components satisfy Killing's equations (for a derivation, see for example [37]):

$$\frac{\partial K^i{}_A}{\partial x^k} g_{il}(\mathbf{r}) + \frac{\partial K^i{}_A}{\partial x^l} g_{ik}(\mathbf{r}) + \frac{\partial g_{kl}}{\partial x^i} K^i{}_A = 0 \, . \tag{14}$$

The algebra (12) is satisfied by

$$(T_k)^i{}_j = -\mathrm{i}\epsilon^i{}_{jk} \tag{15}$$

so the transformation of x^i becomes

$$x^i \to x'^i = x^i + \omega^k \epsilon^i{}_{jk} x^j + \mathcal{O}(\omega^k)^2 \,. \tag{16}$$

Thus, for example, a rotation with only ω^1 non-zero is a rotation about the x-axis. The Killing vectors are then

$$\mathbf{K}_i = \epsilon_i{}^j{}_k x^k \hat{\mathbf{e}}_j \,. \tag{17}$$

Note also that if we make the standard replacements

$$\hat{\mathbf{e}}_i \to \frac{\partial}{\partial x^i} \tag{18}$$

these become the quantum angular momentum operators, up to a factor of i. These have the same commutation relations as T_i.

In general, if we write

$$R = \mathrm{e}^{\mathrm{i}\omega^i T_i} = \mathrm{e}^{\mathrm{i}\omega n^i T_i} \tag{19}$$

where ω is the modulus of ω^i -

$$\omega \equiv [(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2]^{\frac{1}{2}} \tag{20}$$

- and n^i is the associated unit vector

$$n^i \equiv \frac{\omega^i}{\omega} \,, \tag{21}$$

then R represents a rotation through an angle ω about an axis $n^i \hat{\mathbf{e}}_i$.

We can use the transformation law for a vector to put the Killing vectors into spherical polars. The variations in the polar coordinates under rotations are then found to be

$$\theta \to \theta' = \theta + \omega^1 \sin\phi - \omega^2 \cos\phi \tag{22}$$

and

$$\phi \to \phi' = \phi + \omega^1 \cot\theta \cos\phi + \omega^2 \cot\theta \sin\phi - \omega^3 \,. \tag{23}$$

These are non-linear in the coordinates, except when restricted to the subgroup of rotations about the z-axis. Of course, the components of the infinitesimal displacement vector $\mathrm{d}\mathbf{r}$ still transform linearly; for example

$$\mathrm{d}\theta \to \mathrm{d}\theta' = \mathrm{d}\theta + \omega^1 \cos\phi \; \mathrm{d}\phi + \omega^2 \sin\phi \; \mathrm{d}\phi \,. \tag{24}$$

Now for a small rotation producing a small variation in x^i we can see from (16) that

$$x^l \epsilon_{il}{}^m \delta x^i = \delta\omega^k x^l x^j (\delta_{lj}\delta_k^m - \delta_j^m \delta_{lk}) \tag{25}$$

$$= x^l x_l \delta\omega^m - x^m x_k \delta\omega^k \,. \tag{26}$$

There are an infinite number of rotations which produce this variation, each with a different axis, tracing out part of a circle on the surface of a sphere of radius $(x^l x_l)^{\frac{1}{2}}$. The largest such circle is a 'great circle', with the same radius as the sphere and an axis orthogonal to $\mathbf{r} = x^i \hat{\mathbf{e}}_i$, so

$$\delta\omega^m = \frac{x^l \epsilon_{il}{}^m}{x^j x_j} \delta x^i \, . \tag{27}$$

If we were to put a classical point mass at $\mathbf{r}$, this would be a rotation for which no torque is required. (Note the close relation between the coefficients of $\delta\omega^k$ in the right hand side of (25) and the product of inertia tensor[2].) For such rotations, we can define an inverse Killing vector K^{-1} by

$$\delta\omega^i = (K^{-1})^i{}_j \delta x^j \, . \tag{28}$$

transforming under coordinate transformations as a covariant vector on the j index. This can easily be seen to be the matrix inverse of K.

3 The S^2 field space

We now consider the situation where we have two spacetime-dependent fields which transform as the coordinates on an S^2 field space. This space can be embedded in an $\mathbb{R}^3$ field space by associating every point on the S^2 space with a point a fixed distance from the origin of $\mathbb{R}^3$. If a triplet on the $\mathbb{R}^3$ field space is written M^i, then the S^2 field space consists of field states satisfying

$$(M^1)^2 + (M^2)^2 + (M^3)^2 = (M)^2 \, . \tag{29}$$

(This is a generalisation of the usual O(3) sigma model[23] where the fixed distance is unity.) A change of coordinates represents a different choice of field variables; with spherical polars on the $\mathbb{R}^3$ space, this condition becomes

$$r = M \, . \tag{30}$$

θ and ϕ are therefore field variables on (a coordinate patch[37, 38] of) the S^2 field space. The tangent plane to the two-sphere in $\mathbb{R}^3$ is spanned by $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$. $\hat{\mathbf{e}}_r$ is always orthogonal to this plane. The metric on S^2 for these coordinates is just the angular part of (8) with r replaced by M.

We can define 'projective coordinates' π^a $(a = 1, 2)$ for the whole of the Northern hemisphere by projecting the hemisphere down onto the equatorial plane. The rule for the embedding is then that the point (π^1, π^2) in S^2 is associated with the point

$$(M^1, M^2, M^3) = (\pi^1, \pi^2, [(M)^2 - (\pi^1)^2 - (\pi^2)^2]^{\frac{1}{2}}) \tag{31}$$

in $\mathbb{R}^3$. The tangent plane to S^2 is then spanned by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, which are not always orthogonal to each other (for example, on the line of 45° longitude). We

[2] Both Tsukanov[17] and Kuipers[33] analyse classical rotations in a similar manner.

can use the transformation law for covariant vectors to find these in terms of the vielbein $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$:

$$\hat{\mathbf{e}}_1^{(\mathrm{Rp})} = \hat{\mathbf{e}}_x - \frac{M^1}{M^3}\hat{\mathbf{e}}_z \,, \tag{32}$$

$$\hat{\mathbf{e}}_2^{(\mathrm{Rp})} = \hat{\mathbf{e}}_y - \frac{M^2}{M^3}\hat{\mathbf{e}}_z \,. \tag{33}$$

These are the pullback maps for the embedding. Consequently, a vector of S^2

$$V^a \hat{\mathbf{e}}_a^{(\mathrm{Sp})} = V^1 \hat{\mathbf{e}}_1^{(\mathrm{Sp})} + V^2 \hat{\mathbf{e}}_2^{(\mathrm{Sp})} \tag{34}$$

is associated under the embedding with a vector of $\mathbb{R}^3$

$$V^1 \hat{\mathbf{e}}_1^{(\mathrm{Rp})} + V^2 \hat{\mathbf{e}}_2^{(\mathrm{Rp})} = V^1 \hat{\mathbf{e}}_x + V^2 \hat{\mathbf{e}}_y - (V^1 \frac{M^1}{M^3} + V^2 \frac{M^2}{M^3})\hat{\mathbf{e}}_z \,, \tag{35}$$

so the differential map is

$$(V^1, V^2) \to (V'^1, V'^2, V'^3) = (V^1, V^2, -V^1 \frac{M^1}{M^3} - V^2 \frac{M^2}{M^3}) \,. \tag{36}$$

The metric for projective coordinates is

$$g_{ab} = (\hat{\mathbf{e}}_a^{(\mathrm{Rp})}, \hat{\mathbf{e}}_b^{(\mathrm{Rp})}) = \delta_{ab} + \frac{\pi_a \pi_b}{(M)^2 - (\pi^1)^2 - (\pi^2)^2} \,. \tag{37}$$

SO(3) rotations are isometries of S^2. The variations in the polar coordinates are still given by (22) and (23). For projective coordinates, we can use (16) and (31) to find the variations in the coordinates:

$$\pi'^1 = \pi^1 - [(M)^2 - (\pi^1)^2 - (\pi^2)^2]^{\frac{1}{2}} \omega^2 + \pi^2 \omega^3 \,, \tag{38}$$

$$\pi'^2 = \pi^2 + [(M)^2 - (\pi^1)^2 - (\pi^2)^2]^{\frac{1}{2}} \omega^1 - \pi^1 \omega^3 \,. \tag{39}$$

The components of $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are therefore

$$\begin{aligned} &K^1{}_1 = 0 \,, \qquad K^1{}_2 = -[(M)^2 - (\pi^1)^2 - (\pi^2)^2]^{\frac{1}{2}} \,, \qquad K^1{}_3 = \pi^2 \,, \\ &K^2{}_1 = [(M)^2 - (\pi^1)^2 - (\pi^2)^2]^{\frac{1}{2}} \,, \qquad K^2{}_2 = 0 \,, \qquad K^2{}_3 = -\pi^1 \end{aligned} \tag{40}$$

Again, the transformations are non-linear, except when restricted to the SO(2) subgroup. S^2, unlike $\mathbb{R}^3$, does not have a coordinate system in which they are generally linear.

The embedding can be used to obtain a relation between the Killing vectors and the metric. If we consider a small arc generated on the sphere by a great circle rotation, working in Cartesian coordinates we easily obtain

$$\delta s^2 = \delta_{ij} \delta M^i \delta M^j = \delta_{km} (M)^2 \delta\omega^k \delta\omega^m \,, \tag{41}$$

while in a coordinate system x^a on the two-sphere,

$$\delta s^2 = g_{ab}|_P \, \delta x^a \delta x^b \,. \tag{42}$$

Using (28) for the two-dimensional coordinates, we find

$$g_{ab} = \delta_{km}(M)^2(K^{-1})^k{}_a(K^{-1})^m{}_b \,, \tag{43}$$

which can be inverted to obtain

$$g^{ab} = \delta^{ij}\frac{1}{(M)^2}K^a{}_iK^b{}_j \,. \tag{44}$$

For projective coordinates we find

$$g^{ab} = \delta^{ab} - \frac{\pi^a\pi^b}{(M)^2} \,. \tag{45}$$

It is easy to verify that this is the matrix inverse of (37).

Now the Lagrangian for our field theory may be written

$$\mathcal{L} = \frac{1}{2}\frac{\partial M^i}{\partial x^\mu}\frac{\partial M_i}{\partial x_\mu} \tag{46}$$

subject to the constraint (29), where μ ranges over the indices of whatever spacetime we are considering. By comparison with (41) and (42), we see that this may be rewritten

$$\mathcal{L} = \frac{1}{2}g_{ab}(x^c)\partial_\mu x^a\partial^\mu x^b \tag{47}$$

for any coordinate system x^c on S^2. As the arc length on S^2 is invariant under SO(3) transformations, clearly this is too. With the metric (37) this contains the normal kinetic term for a doublet plus self-interaction terms. We will see how this Lagrangian can arise in Part II, but we note here for future use that by carrying out the appropriate binomial expansion, this can be written as

$$\mathcal{L} = \frac{1}{2}\left[\delta_{ab} + M^{-2}\left(1 + \frac{\pi^c\pi_c}{M^2} + \frac{(\pi^c\pi_c)^2}{M^4} + \ldots\right)\pi_a\pi_b\right]\partial_\mu\pi^a\partial^\mu\pi^b \,. \tag{48}$$

4 The SO(3) manifold

4.1 The ω^i coordinates

SO(3) is a compact manifold on which ω^i can be used as coordinates. Using (15) and the projections operators[39, 40]

$$(P^1)^i{}_j = \frac{1}{2}(\delta^i_j - n^in_j - \mathrm{i}n^k\epsilon^i{}_{jk}) \,, \tag{49}$$

$$(P^2)^i{}_j = \frac{1}{2}(\delta^i_j - n^in_j + \mathrm{i}n^k\epsilon^i{}_{jk}) \,, \tag{50}$$

$$(P^3)^i{}_j = n^in_j \,, \tag{51}$$

we can rewrite R as

$$R^i{}_j = \cos\omega\delta^i_j + (1-\cos\omega)n^in_j + \sin\omega n^k\epsilon^i{}_{jk} \,. \tag{52}$$

Using the linear independence of the tensors in this expression, one can show that in the region $0 \le \omega < \pi$ there are no discontinuities in the ω^i coordinates and every element has unique coordinates.

We define the basis vector $\hat{\mathbf{e}}_i$ at a point R as the tangent to a curve of increasing ω^i. So, for example, at the origin (the identity element):

$$\hat{\mathbf{e}}_1 = \left.\frac{\mathrm{d}(\mathrm{e}^{\mathrm{i}\omega^1 T_1})}{\mathrm{d}\omega^1}\right|_{\omega^1=0} = \mathrm{i}T_1 \tag{53}$$

and likewise for $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$:

$$(\hat{\mathbf{e}}_k)^i{}_j = \mathrm{i}(T_k)^i{}_j = \epsilon^i{}_{jk} \tag{54}$$

so the vector space at the origin is the (antihermitian) Lie algebra so(3). The infinitesimal displacement vector is

$$\mathrm{d}\omega^k \hat{\mathbf{e}}_k = \mathrm{i}\mathrm{d}\omega^k T_k\,. \tag{55}$$

Away from the origin,

$$\hat{\mathbf{e}}_j(\omega^i) = \frac{\partial R}{\partial \omega^j}\,, \tag{56}$$

$$\mathrm{d}\omega^i \hat{\mathbf{e}}_k = \mathrm{d}\omega^k \frac{\partial R}{\partial \omega^k} = \mathrm{d}\mathbf{R}\,. \tag{57}$$

We use the Killing form as the inner product on the tangent space at the origin:

$$(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2}\,\mathbf{tr}(\mathbf{XY}) \tag{58}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vectors of the (antihermitian) Lie algebra[19, 41, 42]. (For SO(3), $\mathbf{ad}(\mathbf{X}) = \mathbf{X}$.) This makes the basis an orthonormal one. We will not define an inner product away from the origin at this stage.

The action of an element $g = \mathrm{e}^{\mathrm{i}\phi^k T_k}$ of SO(3) on an element R is:

$$R = \mathrm{e}^{\mathrm{i}\omega^i T_i} \to R' = gR(\omega^i) = \mathrm{e}^{\mathrm{i}\phi^k T_k}\mathrm{e}^{\mathrm{i}\omega^i T_i} = \mathrm{e}^{\mathrm{i}\omega'^i(\omega^j,\phi^k)T_i}\,. \tag{59}$$

A curve starting at R is mapped to a curve starting at R' and a vector at R is mapped to a vector at R'. For example,

$$\mathrm{d}\mathbf{R} \to \mathrm{d}\mathbf{R}' = gR(\omega^i + \mathrm{d}\omega^i) - gR(\omega^i) \tag{60}$$

$$= R'(\omega^i + \mathrm{d}\omega^i) - R'(\omega^i) \tag{61}$$

$$= \mathrm{e}^{\mathrm{i}(\omega'^i + \mathrm{d}\omega'^i)T_i} - \mathrm{e}^{\mathrm{i}\omega'^i T_i} \tag{62}$$

$$= \mathrm{d}\omega'^j \frac{\mathrm{e}^{\mathrm{i}(\omega'^i + \mathrm{d}\omega'^i)T_i} - \mathrm{e}^{\mathrm{i}\omega'^i T_i}}{\mathrm{d}\omega'^j} \tag{63}$$

$$= \mathrm{d}\omega'^j \lim_{\delta\omega'^j \to 0} \frac{\mathrm{e}^{\mathrm{i}(\omega'^i + \delta\omega'^i)T_i} - \mathrm{e}^{\mathrm{i}\omega'^i T_i}}{\delta\omega'^j} \tag{64}$$

$$= \mathrm{d}\omega'^j \frac{\partial R}{\partial \omega'^j} \tag{65}$$

$$= \mathrm{d}\omega'^j\ \hat{\mathbf{e}}_j\big|_{R'} \tag{66}$$

where

$$\mathrm{d}\omega'^i = \frac{\partial \omega'^i}{\partial \omega^j}\mathrm{d}\omega^j \,. \qquad (67)$$

At this point, we could assume that the action of SO(3) on itself is an isometry of the manifold and use this to define a metric for each point on the manifold[19], but we shall not do this, because, as we shall see in Section 5, the most useful definition of the inner product for us is, perhaps surprisingly, *not* one for which SO(3) transformations are isometries of this manifold.

4.2 Standard coordinates

Consider the subgroup

$$H \;=\; \{h = \mathrm{e}^{\mathrm{i}\omega^3 T_3} \quad \forall\, \omega^3\}\,. \qquad (68)$$

This can be used to partition the manifold into cosets

$$L(\theta^a)H = \mathrm{e}^{\mathrm{i}\theta^a T_a}\{\mathrm{e}^{\mathrm{i}\theta^3 T_3} \quad \forall\, \theta^3\}\,. \qquad (69)$$

This means that each element $\mathrm{e}^{\mathrm{i}\omega^i T_i}$ may be written as[3, 5, 6]

$$\mathrm{e}^{\mathrm{i}\omega^i T_i} = \mathrm{e}^{\mathrm{i}\theta^a T_a}\mathrm{e}^{\mathrm{i}\theta^3 T_3} \qquad (70)$$

with unique values of the θ^i, that is the θ^i are a non-degenerate set of coordinates for a large neighbourhood of the origin[3]. Furthermore, they have the same origin as the ω^i coordinates and are the same to first order in a power series expansion of the exponentials. At a point where $\theta^i = \alpha^i$, the tangent vectors to the curves of increasing θ^i are

$$\hat{\mathbf{e}}_1 \;=\; \left.\frac{\partial L}{\partial \theta^1}\right|_{\theta^a=\alpha^a} \mathrm{e}^{\mathrm{i}\alpha^3 T_3}\,, \qquad (71)$$

$$\hat{\mathbf{e}}_2 \;=\; \left.\frac{\partial L}{\partial \theta^2}\right|_{\theta^a=\alpha^a} \mathrm{e}^{\mathrm{i}\alpha^3 T_3}\,, \qquad (72)$$

$$\hat{\mathbf{e}}_3 \;=\; L(\alpha^a)\left.\frac{\mathrm{d}}{\mathrm{d}\theta^3}(\mathrm{e}^{\mathrm{i}\theta^3 T_3})\right|_{\theta^3=\alpha^3} = L(\alpha^a)\mathrm{e}^{\mathrm{i}\alpha^3 T_3}(\mathrm{i}T_3)\,. \qquad (73)$$

At the origin, these reduce to (54), where we use the inner product (58).

[3]Note that there is only one point in the manifold for which the coordinates ω^i cannot be stated uniquely, the point for which $\omega = \pi$. In the θ^i coordinates, on the other hand, any point for which $(\theta^1)^2 + (\theta^2)^2 = \pi^2$ is degenerate. This reflects the topological differences between SO(3) and SO(3)/ SO(2) $\times$ SO(2) (see footnote 18 of [26] and references therein). This is one of many common ways of factorising a rotation matrix - for details of how to carry out such factorisations, see for example Kuipers[33].

4.3 The Involutive Automorphism

Here we recap some theory that will be useful later. If we define

$$\tilde{T}_1 = -T_1\,, \qquad \tilde{T}_2 = -T_2\,, \qquad \tilde{T}_3 = T_3 \tag{74}$$

we find that these also satisfy (12). We therefore define the linear operator $\tilde{}$ which performs an automorphic mapping[5, 6, 41] on the Lie algebra:

$$\tilde{}: \mathrm{i}\omega^i T_i \longrightarrow \mathrm{i}\omega^i \tilde{T}_i\,. \tag{75}$$

This acts as a rotation of the Lie algebra through π about the z-axis. It also induces a mapping on the group manifold. Following Isham[6], the subspace for which $\theta^3 = 0$ (or equivalently $\omega^3 = 0$) is denoted P; elements of P are then inverted by this mapping:

$$\tilde{}: L = \mathrm{e}^{\mathrm{i}\theta^a T_a} \longrightarrow \tilde{L} = \mathrm{e}^{-\mathrm{i}\theta^a T_a} = L^{-1}\,. \tag{76}$$

Elements of the subgroup H, on the other hand, are invariant. Using the BCH identity, the commutation relations for the $\tilde{T}_i$ and the linearity of the operator, one can show that for any two group elements g_1 and g_2,

$$\widetilde{(g_1 g_2)} = \tilde{g}_1 \tilde{g}_2\,. \tag{77}$$

5 The SO(3)/SO(2) manifold

5.1 Diffeomorphisms, embeddings and tangent spaces

The cosets (69) span the coset space SO(3)/SO(2), diffeomorphic to both S^2 and P. This allows 'standard' or 'normal' coordinates θ^a, or alternatively polar or projective coordinates, to be used on all three manifolds.

On the S^2 field space, all three coordinates systems share the same origin (vacuum point), the North pole. The standard coordinates of a point on S^2 are the parameters of the rotation which maps the origin to this point; being field variables, the SO(3) rotation is a local transformation[26]. The isotropy (invariance) group of the vacuum is

$$H(x) = \{h(x) = \mathrm{e}^{\mathrm{i}\theta^3(x)T_3}\} \tag{78}$$

hence the coset LH (or any element of it) maps the origin to the same point as $L \in P \subset \mathrm{SO}(3)$.

We next seek the relations between standard coordinates and our other two coordinates systems. Any rotation $L(\theta^1, \theta^2)$ traces out a line of longitude, so the angle of rotation is the polar coordinate θ:

$$\theta = [(\theta^1)^2 + (\theta^2)^2]^{\frac{1}{2}}\,. \tag{79}$$

From the trigonometry of the equatorial plane,

$$n^1 = \sin\phi \quad \Rightarrow \theta^1 = \theta \sin\phi\,, \tag{80}$$
$$n^2 = -\cos\phi \quad \Rightarrow \theta^2 = -\theta \cos\phi\,. \tag{81}$$

For the projective coordinates, we act with L in the form

$$L^i{}_j = \cos\theta\delta^i_j + (1-\cos\theta)n^i n_j + \sin\theta n^k \epsilon^i{}_{jk} \tag{82}$$

on the vacuum in Cartesian coordinates

$$O^* = \begin{pmatrix} 0 \\ 0 \\ M \end{pmatrix} \tag{83}$$

to find

$$\pi^a = M^a = M\sin\theta n^b \epsilon^a{}_{3b} \,, \tag{84}$$
$$M^3 = L^3{}_3 M = M\cos\theta \,. \tag{85}$$

We can now embed any of the two-dimensional manifolds in $\mathbb{R}^3$ or SO(3). Using standard coordinates, we embed the coset space in SO(3) by choosing a gauge for $H(x)$ [26]:

$$E_1 : L(\theta^1,\theta^2)H \rightarrow L(\theta^1,\theta^2) \in \mathrm{SO}(3) \tag{86}$$

with P the image of this map[4]. Similarly, S^2 is the image of the embedding of P and SO(3)/SO(2) in $\mathbb{R}^3$:

$$E_2 : L^i{}_j(\theta^1,\theta^2) \rightarrow M^i = L^i{}_j(\theta^1,\theta^2)O^{*j} \,, \tag{87}$$

$$E_3 : L^i{}_j(\theta^1,\theta^2)H \rightarrow M^i = (L^i{}_j(\theta^1,\theta^2)HO^*)^i = L^i{}_j(\theta^1,\theta^2)O^{*j} \,. \tag{88}$$

This just leaves the embedding of S^2 in SO(3):

$$E_4 : (\theta^1,\theta^2) \rightarrow L^i{}_j(\theta^1,\theta^2) = \cos\theta\delta^i_j + (1-\cos\theta)n^i n_j + \sin\theta n^k \epsilon^i{}_{jk} \,. \tag{89}$$

For SO(3)/SO(2), the basis vector $\hat{\mathbf{e}}_a$ at a point $L(\theta^1,\theta^2)H$ is defined to be the tangent to a curve of increasing θ^a. So at the origin, (the subgroup H), the basis vectors and infinitesimal displacement vector are

$$\hat{\mathbf{e}}_a = \left.\frac{\mathrm{d}(\mathrm{e}^{\mathrm{i}\theta^b T_b})}{\mathrm{d}\theta^a}\right|_{\theta^1,\theta^2=0} H = \mathrm{i}T_a H \,, \tag{90}$$

$$\mathrm{d}\theta^a \left.\hat{\mathbf{e}}_a\right|_H = \mathrm{i}\mathrm{d}\theta^a T_a H \tag{91}$$

and away from the origin,

$$\hat{\mathbf{e}}_a(\theta^b) = \frac{\partial L}{\partial\theta^a} H \,, \tag{92}$$

$$\mathrm{d}\theta^a \hat{\mathbf{e}}_a = \mathrm{d}\theta^a \frac{\partial L}{\partial\theta^a} H = \mathrm{d}\mathbf{L}H \,. \tag{93}$$

[4] Another way of looking at this is to view LH as a closed curve in SO(3), then E_1 maps it to the point where it intercepts P. This is not a useful point of view in this context, but may be useful when considering homotopy groups[43].

The pullback map induced by E_1 is

$$E_1^* : \hat{\mathbf{e}}_a^{(\mathrm{Cs})}\Big|_{(\theta^b=\alpha^b)} = \frac{\partial L}{\partial \theta^a}\Big|_{\theta^b=\alpha^b} H \to \hat{\mathbf{e}}_a^{(\mathrm{Gs})}\Big|_{(\theta^b=\alpha^b,\theta^3=0)} = \frac{\partial L}{\partial \theta^a}\Big|_{\theta^b=\alpha^b} \tag{94}$$

mapping the vectors (92) to the vectors (71) and (72) on the surface P. The corresponding differential map is

$$E_{1*} : (\mathbf{X}^1, \mathbf{X}^2) \to (\mathbf{X}'^1, \mathbf{X}'^2, \mathbf{X}'^3) = (\mathbf{X}^1, \mathbf{X}^2, 0)\,. \tag{95}$$

Clearly, the embedding associates $\mathrm{d}\mathbf{L}H$ with $\mathrm{d}\mathbf{L}$.

By extending (87)-(88) to the action of curves in P and $\mathrm{SO}(3)/\,\mathrm{SO}(2)$ on O^*, we find the action of the tangent vectors. For example, the action of the tangent vector $\mathbf{X}$ to the curve $L(\lambda)$ is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(L^i{}_j O^{*j})\Big|_{\lambda=0} = \frac{\mathrm{d}L^i{}_j}{\mathrm{d}\lambda}\Big|_{\lambda=0} O^{*j} = \mathbf{X}^i{}_j O^{*j} \tag{96}$$

- that is, the embedding E_2 induces a map from the vector $\mathbf{X}$ tangent to P to a vector tangent to $S^2 \subset R^3$. Similarly E_3 induces a map from $\mathbf{X}H$ to $\mathbf{X}O^*$. In particular, the pullback maps are

$$E_2^* : \hat{\mathbf{e}}_a^{(\mathrm{Gs})}(\theta^a) \to \hat{\mathbf{e}}_a^{(\mathrm{Ss})}(\theta^a) = \frac{\partial M^i}{\partial \theta^a}\hat{\mathbf{e}}_i^{(\mathrm{RC})} = \frac{\partial L^i{}_j}{\partial \theta^a} O^{*j}\hat{\mathbf{e}}_i^{(\mathrm{RC})}\,, \tag{97}$$

$$E_3^* : \hat{\mathbf{e}}_a^{(\mathrm{Cs})}(\theta^a) \to \hat{\mathbf{e}}_a^{(\mathrm{Ss})}(\theta^a) = \frac{\partial M^i}{\partial \theta^a}\hat{\mathbf{e}}_i^{(\mathrm{RC})} = \frac{\partial L^i{}_j}{\partial \theta^a} O^{*j}\hat{\mathbf{e}}_i^{(\mathrm{RC})}\,. \tag{98}$$

At the origin, using (96) and (54) or (90), these give us the basis vectors for standard coordinates on the two-sphere:

$$\hat{\mathbf{e}}_1^{(\mathrm{Ss})} = (\mathrm{i}T_1)^i{}_j O^{*j}\hat{\mathbf{e}}_i^{(\mathrm{RC})} = \epsilon^i{}_{31} M \hat{\mathbf{e}}_i^{(\mathrm{RC})} = M\hat{\mathbf{e}}_y\,, \tag{99}$$

$$\hat{\mathbf{e}}_2^{(\mathrm{Ss})} = (\mathrm{i}T_2)^i{}_j O^{*j}\hat{\mathbf{e}}_i^{(\mathrm{RC})} = \epsilon^i{}_{32} M \hat{\mathbf{e}}_i^{(\mathrm{RC})} = -M\hat{\mathbf{e}}_x\,. \tag{100}$$

The orthonormality of the Cartesian basis then implies that the metric for standard coordinates is flat at the North pole.

The infinitesimal displacement vector at H, (91), or the corresponding vector at the origin of P induce the infinitesimal displacement vector in standard coordinates at the North pole of the two-sphere:

$$\mathrm{d}\theta^a\hat{\mathbf{e}}_a^{(\mathrm{Ss})} = (\mathrm{i}\mathrm{d}\theta^a T_a)^i{}_j O^{*j}\hat{\mathbf{e}}_i^{(\mathrm{RC})} = M\mathrm{d}\theta^1\hat{\mathbf{e}}_y - M\mathrm{d}\theta^2\hat{\mathbf{e}}_x\,. \tag{101}$$

Similarly, the vector $\mathrm{d}\mathbf{L}$ or $\mathrm{d}\mathbf{L}H$ induces the infinitesimal displacement vector at LO^*

$$\mathrm{d}\theta^a\hat{\mathbf{e}}_a^{(\mathrm{Ss})} = \mathrm{d}\theta^a\frac{\partial L^i{}_j}{\partial \theta^a} O^{*j}\hat{\mathbf{e}}_i^{(\mathrm{RC})} = \mathrm{d}\theta^a\frac{\partial M^i}{\partial \theta^a}\hat{\mathbf{e}}_i^{(\mathrm{RC})}\,. \tag{102}$$

By equating this to $\mathrm{d}M^i\hat{\mathbf{e}}_i$ we get the expected differential map

$$\mathrm{d}M^i = \frac{\partial M^i}{\partial \theta^a}\mathrm{d}\theta^a\,. \tag{103}$$

We now wish to find these derivatives, which would make it possible to obtain explicit forms for the pullback map, the infinitesimal displacement vector and the metric. We start by noting that a rotation

$$L(\theta'^a) = L(\theta^a + \mathrm{d}\theta^a) \tag{104}$$

maps O^* to a point with Cartesian coordinates

$$M'^a = M \sin\theta' n'^b \epsilon^a{}_{3b}\,, \tag{105}$$

$$M'^3 = M \cos\theta'\,. \tag{106}$$

We then Taylor expand (79) around θ to obtain

$$\theta' = \theta + n_a \mathrm{d}\theta^a \tag{107}$$

and use

$$\theta'^a = \theta' n'^a \tag{108}$$

to obtain

$$n'^a = n^a + \frac{1}{\theta}(\mathrm{d}\theta^a - n^a n_b \mathrm{d}\theta^b) \tag{109}$$

where $n_a \equiv n^a$. Substituting these into (105) and (106), we find

$$\frac{\partial M^a}{\partial \theta^c} = M\left(\cos\theta \epsilon^a{}_{3b} n^b n_c + \frac{\sin\theta}{\theta}\epsilon^a{}_{3b}(\delta^b_c - n^b n_c)\right)\,, \tag{110}$$

$$\frac{\partial M^3}{\partial \theta^c} = -M \sin\theta n_c \tag{111}$$

(note the appearance of the tensors in P^1, P^2 and P^3[32, 40]). The resulting forms of the basis vectors and the infinitesimal displacement vector, obtained using (97) and (102), are long expressions which are not of much use to us. The crucial quantity is the metric (induced by E_2), which turns out to be quite simple:

$$g^{(\mathrm{Ss})}_{ab} = (\hat{\mathbf{e}}_a, \hat{\mathbf{e}}_b) = (M)^2\left(n_a n_b + \frac{\sin^2\theta}{\theta^2}(\delta_{ab} - n_a n_b)\right)\,. \tag{112}$$

This reduces to $(M)^2\delta_{ab}$ at the origin, while (37) reduces to δ_{ab}. This difference is due to the factor of M in (84) and hence (110).

Finally, as we already have a metric for Cartesian coordinates on $\mathbb{R}^3$, the embeddings E_1, E_2, E_3 can be used to define an inner product at an arbitrary point on both $\mathrm{SO}(3)/\mathrm{SO}(2)$ and P. If we define

$$(\mathbf{X}H, \mathbf{Y}H)|_{LH} \equiv (\mathbf{X}, \mathbf{Y})|_L \equiv \mathbf{X}^i{}_3 \mathbf{Y}_{i3}\,, \tag{113}$$

then using (83) and the orthonormality of the Cartesian basis in $\mathbb{R}^3$,

$$(\mathbf{X}O^*, \mathbf{Y}O^*)|_{LO^*} = (M)^2\,(\mathbf{X}, \mathbf{Y})|_L = (M)^2\,(\mathbf{X}H, \mathbf{Y}H)|_{LH}\,. \tag{114}$$

It is easy to see that (113) reduces to (58) if one bears in mind that at the origin, all vectors tangent to P are linear sums of $\mathrm{i}(T_a)^i{}_j = \epsilon^i{}_{ja}$.

5.2 The action of SO(3)

The actions of an element of SO(3) on P and SO(3)/ SO(2) corresponding to (59) are

$$g : L = \mathrm{e}^{\mathrm{i}\theta^a T_a} \quad \rightarrow \quad gL = L'h \equiv \mathrm{e}^{\mathrm{i}\theta'^a(\theta^b,\phi^i)T_a}\mathrm{e}^{\mathrm{i}\eta^3(\theta^c,\phi^j)T_3}\,, \tag{115}$$

$$g : LH = \mathrm{e}^{\mathrm{i}\theta^a T_a}\{\mathrm{e}^{\mathrm{i}\theta^3 T_3} \quad \forall\, \theta^3\} \quad \rightarrow \quad gLH = L'H \equiv \mathrm{e}^{\mathrm{i}\theta'^a(\theta^b,\phi^i)T_a}\{\mathrm{e}^{\mathrm{i}\theta'^3 T_3} \quad \forall\, \theta'^3\}\,. \tag{116}$$

This implies that g does not commute with E_1 or E_4. It does, however, commute with E_2 and E_3.

The next stage is to find the transformation of the standard coordinates, i.e find θ'^a and η^3. We sketch in outline how this is done[5]. For $g \in H$, it is known[5] that

$$h = g \Rightarrow \eta^3 = \phi^3 \tag{117}$$

and

$$L' = hLh^{-1}\,. \tag{118}$$

We use L in the form (82) and expand h and h^{-1} in powers of ϕ^3 to get

$$L' = L + (1-\cos\theta)(n^i\epsilon_{jkl}n^k\phi^l + \epsilon^i{}_{kl}n^k\phi^l n_j) - \sin\theta\epsilon^m{}_{kl}n^k\phi^l\epsilon^i{}_{mj} + \mathcal{O}(\phi^3)^2\,. \tag{119}$$

We can Taylor expand L' with respect to θ and n^a to first order[6]:

$$\begin{aligned} L' &= L + (-\sin\theta\delta^i_j + \sin\theta n^i n_j - \cos n^k\epsilon^i{}_{kj})\delta\theta \\ &\quad + (1-\cos\theta)(\delta n^i n_j + n^i\delta n_j) - \sin\theta\delta n^m\epsilon^i{}_{kj}\,. \end{aligned} \tag{120}$$

We now equate the traces of (119) and (120). The variation in L in (119) is traceless, so this gives us

$$\delta\theta = 0\,, \qquad n^i\delta n_i = 0\,. \tag{121}$$

The latter is to be expected: the infinitesimal variation in any unit vector in a plane is always orthogonal to the unit vector. We substitute the former into (120) and a comparison with (119) immediately yields

$$\delta n^i = \epsilon^i{}_{kl}n^k\phi^l\,, \tag{122}$$

i.e.

$$\theta'^a = \theta(n^a + \epsilon^a{}_{b3}n^b\phi^3) = \theta^a + \phi^3\theta^b\epsilon^a{}_{b3}\,. \tag{123}$$

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[5] While we write out the indices expicitly here, the author has in practice found it clearer to use the operator notation of Michel and Radicati[42] in which the indices are hidden.

[6] In general, we have to be careful Taylor expanding with respect to n^a, as they are not independent variables, but this does not affect the current calculation.

Note that we have calculated the variations in the standard coordinates on the two-sphere using the properties of the group manifold, and that they transform as a doublet of H. We can check these agree with our earlier results. Using (31), (87) and (119), we find

$$\pi'^1 = M'^1 = \pi^1 + \phi^3\pi^2 \,, \tag{124}$$

$$\pi'^2 = M'^2 = \pi^2 - \phi^3\pi^1 \tag{125}$$

in agreement with (38) and (39). For polar coordinates, we already know that $\delta\theta = 0$. By using (80) and (122) and either considering the change in $\sin\phi$ with a small variation in ϕ or equivalently Taylor expanding ϕ as a function of n^1 (or doing the same with (81), $\cos\phi$ and n^2), we find

$$\delta\phi = -\phi^3 \tag{126}$$

in agreement with (23).

For $g \in P$, the automorphism (76) implies that[5]

$$L'^2 = \mathrm{e}^{2\mathrm{i}\theta'^a T_a} = gL^2 g = g\mathrm{e}^{2\mathrm{i}\theta^a T_a} g \,. \tag{127}$$

This allows us to procede in a similar manner to the case for $g \in H$. We find the variation in L^2 to first order by expanding g in powers of ϕ^a and then Taylor expand L^2 in terms of θ and n^a in order to find the variation in these. The difference this time is that the variation in L^2 is not traceless, so we operate on both expressions for L'^2 with $n^i n_j$ to project out the δn^a parts before equating the traces. Once again, we find

$$n^a \delta n_a = 0 \tag{128}$$

and eventually we get

$$\delta n^a = \cot\theta(\phi^a - n^a n_b \phi^b) = \phi^b \cot\theta(\delta^a_b - n^a n_b) \,. \tag{129}$$

The δn^a parts can now be eliminated from both expressions for L'^2. By using an appropriate tensor identity (resulting from contracting two ϵs in two different ways), we find

$$\delta\theta = \phi^b n_b \,. \tag{130}$$

The standard coordinates of the new point are then

$$\theta'^a = \theta^a + \phi^b[n^a n_b + \theta\cot\theta(\delta^a_b - n^a n_b)] + \mathcal{O}(\phi^b)^2 \tag{131}$$

- once again, the tensors of P^1, P^2, P^3 are appearing. For small θ, this becomes

$$\theta'^a = \theta^a + \phi^a \,. \tag{132}$$

We can find the variation in L by substituting our results into the Taylor expansion for L. Acting with the resulting L' on O^* and using (85) and (31) gives us

$$\pi'^a = \pi^a - M\phi^b \epsilon^a{}_{b3}[(M)^2 - (\pi^1)^2 - (\pi^2)^2]^{\frac{1}{2}} \tag{133}$$

in agreement with (38) and (39). For spherical polars, substituting (80) and (81) into (130) gives us

$$\theta \to \theta' = \theta + \phi^1 \sin\phi - \phi^2 \cos\phi \tag{134}$$

while for the variation in ϕ we again use a Taylor expansion of ϕ to get

$$\delta\phi = \phi^1 \cot\theta \cos\phi + \phi^2 \cot\theta \sin\phi \tag{135}$$

both in agreement with (22) and (23).

Finally, we want to find η^3 for $g \in P$. This can be done by solving (115) for h and substituting in the Taylor expansion of L' and the power series expansion of g, giving us

$$h = \mathbb{1} + \frac{\partial L}{\partial \theta}\delta\theta L^{-1} + \frac{\partial L}{\partial n^a}\delta n^a L^{-1} - L(\mathrm{i}\phi^a T_a)L^{-1} + \mathcal{O}(\phi^a)^2 \,. \tag{136}$$

The second term is easy to calculate using projection operators, but the third and fourth require rather more work. We end up with

$$\begin{aligned}
\left(\frac{\partial L}{\partial \theta}\delta\theta L^{-1}\right)^i{}_j &= \phi^a n_a n^b \epsilon^i{}_{jb} \,, && (137)\\
\left(\frac{\partial L}{\partial n^a}\delta n^a L^{-1}\right)^i{}_j &= \cot\theta(1-\cos\theta)(\phi^i n_j - n^i \phi_j) \\
&\quad + \cos\theta(\phi^a \epsilon^i{}_{ja} - \phi^a n_a n^b \epsilon^i{}_{jb}) \,, && (138)\\
\left(L(\mathrm{i}\phi^a T_a)L^{-1}\right)^i{}_j &= \cos\theta \phi^a \epsilon^i{}_{ja} + (1-\cos\theta)\phi^a n_a n^b \epsilon^i{}_{jb} \\
&\quad - \sin\theta(\phi^i n_j - n^i \phi_j) && (139)
\end{aligned}$$

so that on applying a few trigonometric identities we get

$$h^i{}_j = \delta^i_j - \tan\left(\frac{\theta}{2}\right)(\phi^i n_j - n^i \phi_j) + \mathcal{O}(\phi^a)^2 \tag{140}$$

$$\Rightarrow h = \mathbb{1} + \mathrm{i}\tan\left(\frac{\theta}{2}\right)\phi^a n^b \epsilon_{ab}{}^3 T_3 + \mathcal{O}(\phi^a)^2 \tag{141}$$

so

$$\eta^3 = \tan\left(\frac{\theta}{2}\right) n^a \phi^b \epsilon_{ab}{}^3 \,. \tag{142}$$

Note this is precisely the result found by Barnes *et al*[32].

Consider the curve $\mathrm{e}^{\mathrm{i}\lambda\alpha^a T_a}$ in P, where α^1, α^2 are constants and λ is the curve parameter. Under $g \in \mathrm{SO}(3)$, it transforms as

$$g(\phi^i) : \mathrm{e}^{\mathrm{i}\lambda\alpha^a T_a} \to \mathrm{e}^{\mathrm{i}\theta'^a(\lambda\alpha^b,\phi^i)T_a}\mathrm{e}^{\mathrm{i}\eta^3(\lambda\alpha^c,\phi^j)T_3} \,. \tag{143}$$

In general, any curve $a(\lambda)$ in P is mapped to a curve $a'(\lambda)$ in SO(3) with a varying 'height' $\eta^3(\lambda)$ above P:

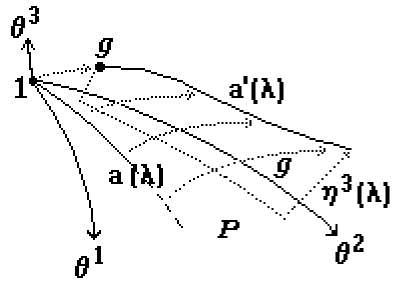

Figure 1. The action of g on a curve $a(\lambda)$ in P.

(In the above diagram, $a(\lambda)$ is shown starting at $\mathbb{1}$, i.e. $a(0) = \mathbb{1}$, but this need not be the case.)

The tangent vector to $a(\lambda)$ at any value of λ, which is tangent to P, is then mapped to an SO(3) vector with components in all three directions. To take a simple example, the basis vector at the origin $\mathrm{i}T_1$ is mapped thus:

$$g : \mathrm{i}T_1 = \left.\frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta^1 T_1}}{\mathrm{d}\theta^1}\right|_{\theta^1=0} \quad \rightarrow \quad \left.\frac{\mathrm{d}}{\mathrm{d}\theta^1}(\mathrm{e}^{\mathrm{i}\theta'^a(\theta^1,\phi^i)T_a}\mathrm{e}^{\mathrm{i}\eta^3(\theta^1,\phi^j)T_3})\right|_{\theta^1=0}$$
$$= c_1\ \hat{\mathbf{e}}_1\big|_g + c_2\ \hat{\mathbf{e}}_2\big|_g + c_3\ \hat{\mathbf{e}}_3\big|_g \qquad (144)$$

where c_1, c_2, c_3 are real coefficients.

A curve in the coset space, on the other hand, is mapped to another curve in the coset space, so a vector tangent to the coset space is mapped to another vector tangent to the coset space. For example,

$$g : \mathrm{i}T_1 H = \left.\frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta^1 T_1}}{\mathrm{d}\theta^1}H\right|_{\theta^1=0} \quad \rightarrow \quad \left.\frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta'^a(\theta^1,\phi^i)T_a}}{\mathrm{d}\theta^1}\right|_{\theta^1=0} H$$
$$= c_1\ \hat{\mathbf{e}}_1\big|_{gH} + c_2\ \hat{\mathbf{e}}_2\big|_{gH} \qquad (145)$$

with the same coefficients c_1, c_2, so g does not commute with E_1^*. In general, the action of a group element on any vector tangent to P is to map it to a vector of SO(3) which has components in all three coordinate directions; the action on the corresponding coset space vector is to map it to a new coset space vector whose components are equal to the first two components of the transformed SO(3) vector.

The most important example of this is the mapping of the infinitesimal displacement vector $\mathrm{d}\mathbf{L}$ at a point L back to the origin. To calculate this, we need $\mathrm{d}\mathbf{L}$ in terms of variations in θ and n^a: from (82),

$$\begin{aligned} \mathrm{d}\mathbf{L} &= -\sin\theta\delta^i_j + (1-\cos\theta)(\mathrm{d}n^i n_j + n^i\mathrm{d}n_j) + \sin\theta n^i n_j \mathrm{d}\theta \\ &\quad + \cos\theta\mathrm{d}\theta n^k\epsilon^i{}_{jk} + \sin\theta\mathrm{d}n^k\epsilon^i{}_{jk} \end{aligned} \qquad (146)$$

where

$$n^a\mathrm{d}n_a = n_a\mathrm{d}n^a = 0\,. \qquad (147)$$

Multiplying this by $L^{-1}(\theta^a) = L(-\theta, n^a)$ gives us

$$(L^{-1}\mathrm{d}\mathbf{L})^i{}_k = (1-\cos\theta)(n^i\mathrm{d}n_k - \mathrm{d}n^i n_k) + \mathrm{d}\theta n^l \epsilon^i{}_{kl} + \sin\theta\cos\theta \mathrm{d}n^j \epsilon^i{}_{kj} + \sin\theta(1-\cos\theta)(n^i\mathrm{d}n^j n^l \epsilon_{jlk} + \epsilon^i{}_{lj} n^l \mathrm{d}n^j n_k) \quad (148)$$

Again, using the contraction of three ϵs, it is easy to show that the last expression in brackets is just $\mathrm{d}n^j\epsilon^i{}_{kj}$, so

$$\begin{aligned}(L^{-1}\mathrm{d}\mathbf{L})^i{}_k &= (1-\cos\theta)\mathrm{d}n^l n^m \epsilon^i{}_{jk}\epsilon^j{}_{lm} + \mathrm{d}\theta n^l\epsilon^i{}_{kl} + \sin\theta \mathrm{d}n^j\epsilon^i{}_{kj} && (149)\\ &= (1-\cos\theta)n^a\mathrm{d}n^b\epsilon_{ab}{}^3(\mathrm{i}T_3)^i{}_k + n^a\mathrm{d}\theta(\mathrm{i}T_a)^i{}_k \\ &\quad + \sin\theta\mathrm{d}n^a(\mathrm{i}T_a)^i{}_k && (150)\end{aligned}$$

which as predicted, has components in all three coordinate directions[7]. It is common to split this into a vector tangent to P at the origin and a vector of the Lie algebra of H:

$$\mathrm{d}\mathbf{a} = (n^a\mathrm{d}\theta + \sin\theta\mathrm{d}n^a)(\mathrm{i}T_a)\,, \quad (151)$$

$$\mathrm{d}\mathbf{v} = (1-\cos\theta)n^a\mathrm{d}n^b\epsilon_{ab}{}^3(\mathrm{i}T_3)\,. \quad (152)$$

As these are orthonormal, the length squared of $L^{-1}\mathrm{d}\mathbf{L}$ is given by

$$(L^{-1}\mathrm{d}\mathbf{L}, L^{-1}\mathrm{d}\mathbf{L})\big|_{\mathbb{1}} = (\mathrm{d}\mathbf{a},\mathrm{d}\mathbf{a})|_{\mathbb{1}} + (\mathrm{d}\mathbf{v},\mathrm{d}\mathbf{v})|_{\mathbb{1}}\,. \quad (153)$$

The action on the corresponding coset space vector is

$$L^{-1} : \mathrm{d}\mathbf{L}H \to \mathrm{d}\mathbf{a}H \quad (154)$$

and the length squared of the resulting vector is

$$(L^{-1}\mathrm{d}\mathbf{L}H, L^{-1}\mathrm{d}\mathbf{L}H)\big|_H = (\mathrm{d}\mathbf{a}H,\mathrm{d}\mathbf{a}H)|_H = (\mathrm{d}\mathbf{a},\mathrm{d}\mathbf{a})|_{\mathbb{1}}\,. \quad (155)$$

Now consider the action of $L^{-1}\mathrm{d}\mathbf{L}$ on O^*. H is the isotropy group of the vacuum, so

$$\mathrm{d}\mathbf{v}\, O^* = 0\,. \quad (156)$$

By using (114) and recalling that SO(3) transformations are isometries of S^2, one can see that the lengths of $\mathrm{d}\mathbf{L}$ and $\mathrm{d}\mathbf{a}$ are equal:

$$\begin{aligned}(M)^2\,(\mathrm{d}\mathbf{L},\mathrm{d}\mathbf{L})|_L &= (\mathrm{d}\mathbf{L}O^*,\mathrm{d}\mathbf{L}O^*)|_{LO^*} && (157)\\ &= (L^{-1}\mathrm{d}\mathbf{L}O^*, L^{-1}\mathrm{d}\mathbf{L}O^*)\big|_{O^*} && (158)\\ &= (\mathrm{d}\mathbf{a}\,O^*,\mathrm{d}\mathbf{a}\,O^*)|_{O^*} && (159)\\ &= (M)^2\,(\mathrm{d}\mathbf{a},\mathrm{d}\mathbf{a})|_{\mathbb{1}}\,, && (160)\end{aligned}$$

but by (153) these are *not* equal to the length of $L^{-1}\mathrm{d}\mathbf{L}$, so with our definitions of inner products, SO(3) transformations are *not* isometries of the SO(3) manifold, but they *are* isometries of the coset space.

[7] This is the Maurer-Cartan form associated with $\mathrm{d}\mathbf{L}$.

The variations generated by SO(3) transformations on the coset space and on the two-sphere may be written in terms of the Killing vectors $\boldsymbol{\mathcal{K}}_i$ and $\mathbf{K}_i$ respectively as:

$$\mathrm{d}\mathbf{L}H = \boldsymbol{\mathcal{K}}_i \mathrm{d}\phi^i = \mathrm{d}\theta^a \frac{\partial L}{\partial \theta^a} H = \mathrm{d}\theta^a \hat{\mathbf{e}}_a^{(\mathrm{Cs})} , \tag{161}$$

$$\mathrm{d}\boldsymbol{\theta} = \mathbf{K}_i \mathrm{d}\phi^i = \mathrm{d}\theta^a \hat{\mathbf{e}}_a^{(\mathrm{Ss})} . \tag{162}$$

Though the bases are different, the components of these equations are the same for both manifolds:

$$\mathrm{d}\theta^a = K^a{}_i \mathrm{d}\phi^i \tag{163}$$

and these can be read off from (123) and (131):

$$K^a{}_3 = \theta^b \epsilon^a{}_{b3} = \mathrm{i}\theta (P^2 - P^1)^a{}_3 , \tag{164}$$

$$K^a{}_b = n^a n_b + \theta \cot\theta (\delta^a_b - n^a n_b) = (P^3)^a{}_b + \theta \cot\theta (P^1 + P^2)^a{}_b . \tag{165}$$

To transform to another coordinate system we just use the transformation law for a vector. Barnes *et al*[32] and Hamilton-Charlton[36] use coordinates proportional to n^a (they call them M^a) and get results equivalent to ours. Using projection operators makes finding the contravariant metric (44) easy to obtain:

$$g^{ab} = \frac{1}{(M)^2} [n^a n^b + \theta^2 \operatorname{cosec}^2 \theta (\delta^{ab} - n^a n^b)] . \tag{166}$$

They also make it easy to invert this and we once again get (112). The arc length in standard coordinates is therefore

$$\mathrm{d}s^2 = (M)^2 \left(n_a n_b + \frac{\sin^2\theta}{\theta^2} (\delta_{ab} - n_a n_b) \right) \mathrm{d}\theta^a \mathrm{d}\theta^b . \tag{167}$$

There is a third way to get this. We know that

$$\mathrm{d}s^2 = (\mathrm{d}\mathbf{M}, \mathrm{d}\mathbf{M})|_M = (\mathrm{d}\mathbf{L}O^*, \mathrm{d}\mathbf{L}O^*)|_{LO^*} = (M)^2 \, (\mathrm{d}\mathbf{a}, \mathrm{d}\mathbf{a})|_{\mathbb{1}} . \tag{168}$$

By using (107), (109) and (151), we find

$$\mathrm{d}\mathbf{a} = \left(n^a n_b \mathrm{d}\theta^b + \tfrac{\sin\theta}{\theta} (\mathrm{d}\theta^a - n_a n_b \mathrm{d}\theta^b) \right) (\mathrm{i}T_a) \tag{169}$$

$$= \left(P^3 + \tfrac{\sin\theta}{\theta} (P^1 + P^2) \right)^a{}_b \mathrm{d}\theta^b (\mathrm{i}T_a) \tag{170}$$

and by substituting this into (168) we again obtain (167). The Lagrangian for the sigma model in standard coordinates is therefore

$$\mathcal{L} = \frac{1}{2} (M)^2 \left(n_a n_b + \frac{\sin^2\theta}{\theta^2} (\delta_{ab} - n_a n_b) \right) \partial_\mu \theta^a \partial^\mu \theta^b . \tag{171}$$

Note that under the coordinate transformations mentioned above, these are equivalent to (69) and (76) of Barnes *et al*; Hamilton-Charlton also finds these results under these coordinate transformations for SO(3) and SU(2) and discusses the homomorphism.

6 Discrete symmetries

The arc length $dM^i dM_i$ is not just invariant under continuous SO(3) transformations, described by Killing vectors, it is also invariant under the discrete transformation

$$M^i \rightarrow -M^i \quad \forall M^i \,, \tag{172}$$

i.e. the full isometry group of the two-sphere is $O(3) = SO(3) \otimes \mathbb{Z}_2$.

The discrete transformation maps each point on the sphere to the opposite point. It cannot be represented in projective coordinates, as they are only valid for one hemisphere. In spherical polar coordinates, ϕ ranges from 0 to 2π while θ ranges from 0 to π and the discrete transformation is represented by

$$\theta \rightarrow \pi - \theta \,, \qquad \phi \rightarrow \phi \pm \pi \tag{173}$$

where the sign is whatever is needed to keep ϕ in the allowed range. For standard coordinates, the modulus θ transforms as above, while the axis of rotation is reversed, so

$$\theta^a = \theta n^a \rightarrow (\pi - \theta)(-n^a) = \theta^a - \frac{\pi\theta^a}{[(\theta^1)^2 + (\theta^2)^2]^{\frac{1}{2}}} \,. \tag{174}$$

It is possible to find an element of P which maps a particular point M^i on the sphere to its opposite point $-M^i$ as follows.

If (82) maps O^* to a point M^i, by applying the above transformation to (82) we see that

$$(L_{opp})^i{}_j = -\cos\theta\delta^i_j + (1 + \cos\theta)n^i n_j - \sin\theta n^k \epsilon^i{}_{jk} \tag{175}$$

maps O^* to the opposite point $-M^i$. It is easy to verify explicitly, particularly if one uses projection operators, that the combined map from M^i to $-M^i$ given by $L_{opp}L^{-1}$ is just a rotation through π about n^a.

However this is, of course, not the same as the discrete transformation: only points on the same line of longitude as M^i are mapped to their opposite points. In general, a rotation through π about any axis orthogonal to the line between M^i and $-M^i$ will map all the points on a great circle connecting them, and only these points, to their opposite points.

One might suspect from (171) that the transformation

$$\theta^a \rightarrow -\theta^a \tag{176}$$

is a discrete isometry of the two-sphere, but it is easy to show that this is just a rotation through π about the z-axis, related to the involution on the group space described in Section 4.3 by the diffeomorphism/embeddings.

7 Local SO(2) invariance

(115) can be seen as two combined transformations: firstly a mapping in P from L to L' and secondly an action from the right with an element of a local SO(2) group. We follow Balachandran *et al*[26] and denote this $H_C(x)$. While it is isomorphic to a local version of $H \subset G$, its action is different:

$$h \in H : M^i = L^i{}_j O^{*j} \to M'^i = L'^i{}_j O^{*j} = (hLh^{-1})^i{}_j O^{*j} \tag{177}$$

while

$$h_C \in H_C : M^i = L^i{}_j O^{*j} \to L^i{}_j h^j{}_k O^{*k} = L^i{}_j O^{*j} \tag{178}$$

so M^i is a singlet of $H_C(x)$. $L^{-1}\partial_\mu \mathbf{L}$, on the other hand, transforms as:

$$h_C(x) : L^{-1}\partial_\mu \mathbf{L} \to h_C^{-1}(L^{-1}\partial_\mu \mathbf{L})h_C + h_C^{-1}\partial_\mu h_C\ . \tag{179}$$

With an Abelian $H(x)$ it is easy to show that the last term is in the Lie subalgebra, so

$$h_C(x) : \mathbf{v}_\mu \quad \to \quad \mathbf{v}'_\mu = h_C^{-1}\mathbf{v}_\mu h_C + h_C^{-1}\partial_\mu h_C\ , \tag{180}$$

$$h_C(x) : \mathbf{a}_\mu \quad \to \quad \mathbf{a}'_\mu = h_C^{-1}\mathbf{a}_\mu h_C \tag{181}$$

i.e. $\mathbf{v}_\mu$ (the subalgebra part of $L^{-1}\partial_\mu \mathbf{L}$) transforms like a gauge field, while $\mathbf{a}_\mu$ (the part tangent to P) transforms as a vector of the adjoint representation of $H_C(x)$. Our Lagrangian is an invariant of $H_C(x)$ (proportional to the length squared of $\mathbf{a}_\mu$ or $\partial_\mu \mathbf{L} O^*$) and looks like the mass term of the adjoint representation field $\mathbf{a}_\mu$ [27].

If we add any bilinear function of $\mathbf{v}_\mu - eA^3_\mu(\mathrm{i}T_3)$ to the Lagrangian, where e is an arbitrary coupling constant, the Euler-Lagrange equation for A^3_μ is

$$A^3_\mu = v^3_\mu \tag{182}$$

which, by (152), is a function of θ^a and $\partial_\mu\theta^a$. If this is to be covariant, $A^3_\mu(\mathrm{i}T_3)$ must transform the same way as $\mathbf{v}$, i.e. it is an auxilliary gauge field. Thus at a classical level, the new Lagrangian is equivalent to the old one[30]. (We will discuss quantum corrections to this in Section 14.) For example,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(M)^2 \left(n_a n_b + \frac{\sin^2\theta}{\theta^2}(\delta_{ab} - n_a n_b) \right) \partial_\mu\theta^a \partial^\mu\theta^b \\
&\quad + f(\theta, M)(\mathbf{v}_\mu - e\mathbf{A}_\mu)^a{}_b \theta^b (\mathbf{v}^\mu - e\mathbf{A}^\mu)_{ac}\theta^c \qquad (183) \\
&= \frac{1}{2}(M)^2 \left(\partial_\mu\theta^a\partial^\mu\theta^a + \left(\frac{\sin^2\theta}{\theta^2} - 1 \right)(\delta_{ab} - n_a n_b)\partial_\mu\theta^a\partial^\mu\theta^b \right) \\
&\quad + f(\theta, M)(\mathbf{v}_\mu - e\mathbf{A}_\mu)^a{}_b \theta^b (\mathbf{v}^\mu - e\mathbf{A}^\mu)_{ac}\theta^c \qquad (184)
\end{aligned}$$

is equivalent at the classical level to (171). By using (107), (109) and (152) and the properties of projection operators, it is not hard to show that

$$(\mathbf{v}_\mu)^a{}_b \theta^b (\mathbf{v}^\mu)_{ac}\theta^c \propto (\delta_{ab} - n_a n_b)\partial_\mu\theta^a\partial^\mu\theta^b \tag{185}$$

so by taking an appropriate function $f(\theta, M)$, we may cancel the non-linear part of (184). (171) is therefore equivalent to a Lagrangian with no non-linear self-couplings of the θ^a but with instead non-linear couplings to an auxilliary gauge field.

8 Classical solutions - Summary of known results

$\partial_\mu \theta^a \hat{\mathbf{e}}_a^{(Ss)}$ is a vector tangent to the field space, the coordinates of which are themselves functions of spacetime. As the field space is curved, one can define a covariant derivative of a vector such as this. By analogy with the three-sphere[7], it is known that the Euler-Lagrange equations of the Lagrangian (171) simply say that

$$\mathrm{D}_\mu \partial^\mu \theta^a = 0 \tag{186}$$

thus describing the parallel transport of $\partial_\mu \theta^a \hat{\mathbf{e}}_a^{(Ss)}$ along a geodesic[19]. It is in exploring the classical solutions of these equations, especially for fields on two-, three- or four-dimensional spacetime, that most attention has focused over the years. This is simpler in cases where the 'effective' spacetime has a positive definite metric - either Euclidean spacetime or static solutions (which can then be Lorentz boosted to give uniformly moving solutions).

In the case where only one dimension has spacelike signature, Pohlmeyer has shown that the soliton is related to the sine-Gordon kink by a Backlund transformation[44]. When two dimensions are spacelike, the only finite action solution with localised topological charge is the instanton[23, 45, 46]. However, a number of infinite action solutions have been shown to have physical relevance - these are summarised by Ody and Ryder[47] who relate them to surfaces of constant mean curvature in $\mathbb{R}^3$. This model has strong connections with the two-dimensional Heisenberg ferromagnet[23], with Euclidean two-dimensional Yang-Mills theories[48] and with Yang-Mills and Yang-Mills-Higgs and O(5)-invariant theories in four-dimensional Minkowski spacetime[27, 45, 49, 50, 51].

The instanton one finds with two spacelike dimensions does not exist when one has three spacelike dimensions, unless the fields are coupled to gravity[26, 50]. The monopole is a static, topologically stable solution, but one can also have unstable 'embedded' strings and domain walls[52]. (These last two papers consider the sigma model in its role as the low-energy effective theory when O(3) is spontaneously broken, as described in Section 11.) These interact in a non-trivial way[53]. Tsukanov has also looked at rotating soliton solutions[17].

9 Summary of Part I

We conclude Part I by summarising what we have done so far. We took an $\mathbb{R}^3$ field space and constrained the fields to lie on the surface of a sphere in this space. We noted that two of the set of spherical polar coordinates were valid coordinates (field variables) for this space and we defined projective coordinates for the Northern hemisphere. There is an SO(3) group of isometries on the manifold, with the North pole left invariant under rotations about the z-axis. We found the metric (and hence the arc length) in both coordinate systems.

The group SO(3) also forms a Riemannian manifold, for which we defined two sets of coordinates and found the corresponding bases on the tangent space. The second set, standard coordinates, allowed us to show that the subspace P is diffeomorphic to both S^2 and SO(3)/ SO(2) and allowed us to define corresponding coordinates on these manifolds. We showed how to embed each of these three spaces in $\mathbb{R}^3$ and SO(3).

We found how the coordinates and elements of the two-dimensional manifolds transform under SO(3) transformations. In particular, we found the variation of the standard coordinates algebraically, without needing to make use of the embedding in $\mathbb{R}^3$. Under transformations in the subgroup H, they transform as a doublet of the subgroup, while under transformations in P they transform non-linearly.

We established a consistent set of inner products for the various spaces and showed that with these inner products, SO(3) transformations are isometries of S^2 and SO(3)/ SO(2) but *not* the SO(3) manifold itself.

The Lagrangian for the constrained field theory is obtained trivially from the expression for the arc length on S^2 and is similarly invariant under SO(3) transformations. We found this in three different ways: by calculating the metric induced by the embedding in $\mathbb{R}^3$, by using the Killing vectors and by calculating the length of the part of $L^{-1}\mathrm{d}\mathbf{L}$ tangent to P.

We looked at the action of the discrete $\mathbb{Z}_2$ symmetry on the S^2 field space and found its action on the coordinates. This action could be replicated with an SO(3) element for a great circle on the sphere but not for the whole space simultaneously.

We demonstrated the invariance of the Lagrangian under local SO(2) transformations and it was shown to be equivalent to that for a doublet with non-linear couplings to an auxilliary gauge field.

Finally, we summarised some known results relating to the solutions of the classical equations of motion of the model in different numbers of dimensions.

This study was greatly aided by the fact that the field spaces may so readily be visualised and as such, the author feels this is a very useful tool for anyone learning the subject. However, the concepts involved can be applied to a wide variety of non-linear field theories - in particular, they can be extended almost trivially to other spherical field spaces, diffeomorphic to $\mathrm{SO}(N)/\,\mathrm{SO}(N-1)$, and to field spaces diffeomorphic to certain subspaces of $\mathrm{SU}(N)$ groups.

In Part II, we shall show how the constrained field space described here results naturally from spontaneously breaking a system with linear, manifest SO(3) symmetry, as the low-energy effective theory. We therefore have, in principle, a method for obtaining the low-energy limit of many symmetry-breaking schemes.

Part II
Extensions of the model

10 Introduction to Part II

It has been said that to have mastered a problem, you must understand it from three different perspectives. In Part I, we demonstrated explicitly the equivalence between two perspectives: the sigma model perspective (from constraining the norm of a scalar multiplet, first considered by Gell-Mann and Lévy[1]) and the non-linear realisation perspective based on the standard coordinates and the coset space representative L[3, 5]. We start Part II by considering a third perspective, that of spontaneous symmetry breaking, famously studied in the case of global symmetries by Goldstone[54]. This makes use of the embedding of the spherical field space into three-dimensional Euclidean space, thereby adding a third Lorentz scalar component. We then go on to look at how this classical, geometrical model may be related to something more like 'real world' physics by introducting new fields such as non-Abelian gauge fields and fermions and looking at quantisation and renormalisation.

Goldstone's model was extended to the case of an arbitrary continuous group G and continuous subgroup H by Salam and Strathdee[18], based on a parametrisation used by Higgs[55] and Kibble[56]. In Section 11 we use the methods of Salam and Strathdee to demonstrate, with explicit calculations, how the low-energy limit of a system with spontaneously broken global $O(3)$ symmetry is precisely the $O(3)$ non-linear sigma model. While their methods are straightforward to apply to our case, the author believes these notes to be the first explicit demonstration of these three perspectives for a particular symmetry breaking. Furthermore, these methods may readily be extended to other symmetry breaking patterns, giving us a much clearer understanding of spontaneous symmetry breaking.[8] We also show that using a field redefinition analogous to that of Goldstone simply amounts to a different choice of coordinates on the Euclidean field space and show how these are related to those of the sigma model.

The concept of spontaneous symmetry breaking is a crucial part of the Standard Model and extensions of the Standard Model are largely based on spontaneously breaking higher-dimensional symmetry groups to those of the Standard Model. However, Electroweak theory[57, 58] and most Grand Unified Theories are based on the Higgs mechanism[55], where G is gauged and upon symmetry

[8] Salam and Strathdee show that whenever a field theory involves a non-linear constraint on a multiplet and it is to be interpreted using a power series expansion, the vacuum must necessarily be degenerate. For this reason we consider spontaneous symmetry breaking, where the potential is manifestly invariant under $O(3)$, rather than adding explicit symmetry breaking terms.

breaking, some of the gauge fields acquire a mass at the expense of eliminating the Goldstone fields.

The first to study the Higgs mechanism for non-Abelian G was Kibble[56] while the coupling of non-linear realisations to gauge fields was studied by Callan *et al*[59] and implemented by Isham[60] in the case of chiral $SU(3) \otimes SU(3)$. Again, it was Salam and Strathdee who showed that the latter is just the low-energy effective theory resulting from the former. Working at the same time, Honerkamp[61] also demonstrated the equivalence of the two for chiral $SU(3) \otimes SU(3)$.[9] We consider the effect of introducing an SO(3) gauge field in Section 12 and again use the methods of Salam and Strathdee to find the broken Lagrangian[10].

Having analysed the classical scalar theories in detail, the remaining sections give a précis of known techniques and results. In Section 13 we look at how fermions and supersymmetry may be incorporated into the models. We start with minimally coupling fermions to the models with global and gauged SO(3). This is described thoroughly in the literature[5, 59, 18], so we just summarise the techniques and basic features.

We then turn to incorporating supersymmetry into our models. We summarise the techniques used in the literature for spontaneously breaking global SO(3) in a model containing chiral superfields only. The importance of the complexification of SO(3) in the supersymmetric model is widely recognised, but there seems to be little explanation of the physical meaning of these transformations, so we describe the effect of various transformations in this group. We also collect together a number of findings and observations made by other authors working on such models. In the case of the gauge theory, we summarise the work of Fayet and point out its use in Seiberg-Witten theory.

Finally, in Section 14 we discuss the quantisation and renormalisation of the models. Since the sigma model was first proposed, a lot work has been done on its quantisation and renormalisation, including on the emergence of topological features and on the use of counterterms. In particular, it is worth noting that at the one-loop level, the low-energy effective theory is equivalent to a theory of a doublet coupled non-linearly to an SO(2) gauge field. This offers the enticing possibility of deriving familiar gauge field-matter systems from scalar self-couplings in a theory with a larger global symmetry, after taking appropriate limits. For the gauged model, we recap its renormalisability properties and also consider the classical spectrum.

[9] Isham[60] also mentions symmetry breaking, but he assumes it to be explicit rather than spontaneous symmetry breaking.

[10] This model is a very old one, most famously described by Georgi and Glashow[62] (although they cite other authors who had previously suggested it), but this formulation of it lends itself to extension to other symmetry-breaking patterns and may tie the earlier sections in with ideas readers are more familiar with. Filk *et al*[63] use similar methods - their work is based on representations of SU(2) on the lattice.

11 Including a radial field - the Goldstone mechanism

11.1 Recap of Part I

We start by recapping the bits of Part I of most immediate use in this section. Five spaces were studied: an $\mathbb{R}^3$ field space and its S^2 subspace, the SO(3) manifold and its subspace P and the coset space SO(3)/ SO(2). For any coordinate system x^a ($a = 1, 2$) on the constrained S^2 field space, the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2} g_{ab}(x^c) \partial_\mu x^a \partial^\mu x^b \tag{187}$$

where g_{ab} is the metric. The three coordinate systems we considered had their origin at the 'North pole', the point in $\mathbb{R}^3$ where the z-axis intercepts the two-sphere:

$$O^* = \begin{pmatrix} 0 \\ 0 \\ M \end{pmatrix} . \tag{188}$$

(M is the radius of the two-sphere - in the usual formulation of the O(3) sigma model[23] this is taken to be unity.) The isotropy group of this vacuum point is an SO(2) subgroup of the O(3) isometry group. Standard coordinates $\theta^1, \theta^2, \theta^3$ were introduced on the SO(3) manifold and it was shown that θ^1, θ^2 are a valid set of coordinates in a neighbourhood of the origin of the coset space SO(3)/ SO(2) and the S^2 field space, as both of these are diffeomorphic to P, the $\theta^3 = 0$ subspace of SO(3). We identified the embeddings of the three two-dimensional spaces into the three-dimensional ones:

$$\begin{aligned}
E_1 : L(\theta^1, \theta^2)H \quad &\rightarrow \quad L(\theta^1, \theta^2) , && (189)\\
E_2 : L^i{}_j(\theta^1, \theta^2) \quad &\rightarrow \quad M^i = L^i{}_j(\theta^1, \theta^2) O^{*j} , && (190)\\
E_3 : L^i{}_j(\theta^1, \theta^2)H \quad &\rightarrow \quad M^i = (L^i{}_j(\theta^1, \theta^2) H O^*)^i = L^i{}_j(\theta^1, \theta^2) O^{*j} , && (191)\\
E_4 : (\theta^1, \theta^2) \quad &\rightarrow \quad L^i{}_j(\theta^1, \theta^2) = \cos\theta \delta^i_j + (1 - \cos\theta) n^i n_j \\
&\qquad + \sin\theta n^k \epsilon^i{}_{jk} && (192)
\end{aligned}$$

(where $i, j, k = 1, 2, 3$). Taking the tensor form of the SO(3) generators to be

$$(T_k)^i{}_j = -\mathrm{i}\epsilon^i{}_{jk} , \tag{193}$$

the general form for L is:

$$L^i{}_j = \cos\theta \delta^i_j + (1 - \cos\theta) n^i n_j + \sin\theta n^k \epsilon^i{}_{jk} \tag{194}$$

with $n^3 = 0$. (190) then gave us the relationship between Cartesian coordinates on $\mathbb{R}^3$ and standard coordinates on the S^2 subspace:

$$\begin{aligned}
M^a &= L^a{}_b O^{*b} + L^a{}_3 O^{*3} = L^a{}_3 M = M \sin\theta n^b \epsilon^a{}_{3b} , && (195)\\
M^3 &= L^3{}_3 M = M \cos\theta && (196)
\end{aligned}$$

and we also found the relations with other coordinates and determined how each set of coordinates tranforms under SO(3) (and indeed O(3)) transformations. We used the above embeddings to define inner products at arbitrary points on P and the coset space:

$$(\mathbf{X}H, \mathbf{Y}H)|_{LH} \equiv (\mathbf{X}, \mathbf{Y})|_{L} \equiv \mathbf{X}^{i}{}_{3}\mathbf{Y}_{i3}\,. \tag{197}$$

We calculated $L^{-1}\mathrm{d}\mathbf{L}$ and split it into a part $\mathrm{d}\mathbf{a}$ tangent to P and a part $\mathrm{d}\mathbf{v}$ orthogonal to it. The spacetime derivative corresponding to the former part has length squared

$$(\mathbf{a}_{\mu}, \mathbf{a}^{\mu})|_{\mathbb{1}} = \left(n_a n_b + \frac{\sin^2\theta}{\theta^2}(\delta_{ab} - n_a n_b)\right)\partial_{\mu}\theta^a\partial^{\mu}\theta^b \tag{198}$$

from which one may derive (187) in standard coordinates:

$$\mathcal{L} = \frac{1}{2}(M)^2\left(n_a n_b + \frac{\sin^2\theta}{\theta^2}(\delta_{ab} - n_a n_b)\right)\partial_{\mu}\theta^a\partial^{\mu}\theta^b\,. \tag{199}$$

11.2 Method and calculation

In this section, we demonstrate the Goldstone mechanism for spontaneously breaking O(3) to SO(2) and show how this relates to the sigma model as described in Part I. We start with a triplet of O(3), M^i. The kinetic term in the Lagrangian again looks like the arc length on field space, while for the potential to be O(3) invariant, it must be a function of M^iM_i:

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}M^i\partial^{\mu}M_i - V(M^iM_i)\,. \tag{200}$$

Firstly, we look at the potential. This could be any function of $M^iM_i = r^2$, but a constant may always be trivially added to ensure that the minima are $V = 0$. They may occur at any values of r; if there is a minimum at $r = 0$ then (200) is a valid Lagrangian for a triplet of dynamic, self-interacting fields. If not, then we must redefine the fields such that the new set all reduce to zero at one of the minima[54] - this is then the true vacuum state. For example, the potential

$$V = \mu^2(M^iM_i - \rho^2)^2 = \mu^2(r^2 - \rho^2)^2 \tag{201}$$

has a local maximum at $r = 0$ and has degenerate minima on a two-sphere of radius ρ. In keeping with our work thus far, we take the North pole to be the true vacuum; however, we still have an infinite number of coordinate systems to choose from with this origin. One well-known choice[31] - analogous to the choice made by Goldstone for the case of SU(2) - is to keep M^1 and M^2 from the original multiplet and to take as the third field

$$\chi = M^3 - \rho \tag{202}$$

thus translating the Cartesian coordinate system up the z-axis to the North pole. With the potential (201), the Lagrangian (200) then becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu M^a \partial^\mu M_a + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \mu^2 (M^a M_a + \chi^2 + 2\rho\chi)^2 . \tag{203}$$

Note that χ has a mass $2\sqrt{2}\mu\rho$, while M^1 and M^2 are massless - these are the 'Goldstone bosons'[54, 64].

An alternative is to use two coordinates on the two-sphere, together with the radial displacement from the two-sphere:

$$r' = r - \rho . \tag{204}$$

With these coordinates, (201) becomes

$$V = \mu^2 (r'^2 + 2\rho r')^2 . \tag{205}$$

This technique of complementing the coordinates on the subspace with an additional field or fields to cover a larger space is used extensively in Kaluza-Klein theory[65]. Salam and Strathdee[18] have also shown it to be a very elegant way of describing spontaneous symmetry breaking. In this case, the subspace is the vacuum manifold, the coordinates on it are Goldstone bosons and the fields orthogonal to it acquire a mass.

Any coordinates may be used on the vacuum manifold, but there are a number of reasons for using the standard coordinates. Firstly, note that in (189), using a subset of the coordinates on SO(3) for SO(3)/ SO(2) makes both the embedding and the induced map on the tangent space very simple. Using r', θ^1, θ^2 as coordinates on $\mathbb{R}^3$ makes E_2 and E_3 and their induced maps become equally simple. Secondly, unlike, for example, θ and ϕ in the spherical polar system, the standard coordinates transform as a representation of H. Thirdly, standard coordinates may be defined for any G and H where G is a compact, connected semisimple Lie group and H is a proper Lie subgroup, and Salam and Strathdee provide a universal technique for re-writing the Lagrangian (200) in terms of standard coordinates and a set of fields orthogonal to the subspace.

In studying the non-linear sigma model, we considered points on a sphere of radius M which could be mapped back to (188) with L^{-1}, allowing us to associate such a point with the coordinates θ^a. However, any point in the $\mathbb{R}^3$ field space lies on a sphere of some radius r centred at the origin. Thus there is an L^{-1} which maps it to a point on the z-axis:

$$L^{-1} : (M^1, M^2, M^3) \to (0, 0, m^3) \tag{206}$$

or

$$M^i = L^i{}_j(\theta^1, \theta^2) m^j \tag{207}$$

where m^j are the Cartesian coordinates of the point on the z-axis. The rotation keeps the distance from the origin constant, so

$$m^3 \equiv r = r' + \rho , \tag{208}$$

so we have a rule for associating M^i with coordinates r', θ^1, θ^2. By comparison with (195) and (196) we find:

$$M^1 = -(r'+\rho)n^2 \sin\theta\,, \tag{209}$$
$$M^2 = (r'+\rho)n^1 \sin\theta\,, \tag{210}$$
$$M^3 = (r'+\rho)\cos\theta\,. \tag{211}$$

The third field in the Goldstone coordinate system is then

$$\chi = (r'+\rho)\cos\theta - \rho\,. \tag{212}$$

For small θ (close to the North pole) these reduce to

$$M^1 = -(r'+\rho)\theta^2\,, \qquad M^2 = (r'+\rho)\theta^1\,, \qquad \chi = r'\,. \tag{213}$$

Under the SO(2) subgroup M^a and θ^a both transform as doublets, while χ and r' transform as singlets. However, the transformation properties under the full O(3) group are different: neither the M^a nor χ form a realisation of SO(3) on their own (i.e. a general SO(3) rotation will mix up all three fields). The θ^a, on the other hand, transform as a complete (non-linear) realisation of O(3) while r' is invariant.

Now we turn to the kinetic part of $\mathcal{L}$. In spherical polar coordinates, this has a term corresponding to the arc length on the two-sphere and a $\partial_\mu r \partial^\mu r$ term - writing the metric on a two-sphere with unit radius g_{ab},

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu r \partial^\mu r + r^2 \partial_\mu\theta\partial^\mu\theta + r^2\sin^2\theta\,\partial_\mu\phi\partial^\mu\phi) \tag{214}$$
$$= \tfrac{1}{2}[\partial_\mu r\partial^\mu r + r^2(g_{ab}\partial_\mu x^a \partial^\mu x^b)]\,. \tag{215}$$

Using the methods of Salam and Strathdee, we can show that in the coordinates r', θ^1, θ^2 this takes a similar form. The length of the vector $\partial_\mu \mathbf{M}$ is invariant under SO(3) rotations, so we rotate it back to the z-axis and use (207) to calculate its length:

$$(\partial_\mu \mathbf{M}, \partial^\mu \mathbf{M})\big|_M = (L^{-1}\partial_\mu \mathbf{M}, L^{-1}\partial^\mu \mathbf{M})\big|_m \tag{216}$$
$$= (\partial_\mu \mathbf{m} + (L^{-1}\partial_\mu \mathbf{L})\mathbf{m}, \partial^\mu \mathbf{m} + (L^{-1}\partial^\mu \mathbf{L})\mathbf{m})\big|_m\,. \tag{217}$$

where the meanings of the points M and m and the column vector $\mathbf{m}$ are obvious. The Euclidean space has a flat metric. $L^{-1}\partial_\mu \mathbf{L}$ may be split into $\mathbf{a}_\mu = a^a_\mu(\mathrm{i}T_a)$ and $\mathbf{v}_\mu = a^3_\mu(\mathrm{i}T_3)$. From (193) we see that

$$(\mathbf{a}^\mu)^3{}_3 = (\mathbf{v}^\mu)^3{}_3 = (\mathbf{v}^\mu)^k{}_3 = 0\,. \tag{218}$$

Then using $m^a = 0$, (208) and finally (197), we find

$$\partial_\mu M^i \partial^\mu M_i = \partial_\mu r' \partial^\mu r' + (r'+\rho)^2 (\mathbf{a}_\mu)^k{}_3 (\mathbf{a}^\mu)_{k3} \tag{219}$$
$$= \partial_\mu r'\partial^\mu r' + (r'+\rho)^2\,(\mathbf{a}_\mu, \mathbf{a}^\mu)\big|_{\mathbb{1}}\,. \tag{220}$$

Using (198) then gives us the expected form

$$\frac{1}{2}\partial_\mu M^i\partial^\mu M_i = \frac{1}{2}\Big[\partial_\mu r'\partial^\mu r' + (r'+\rho)^2\Big(n_a n_b + \frac{\sin^2\theta}{\theta^2}(\delta_{ab} - n_a n_b)\Big)\partial_\mu\theta^a\partial^\mu\theta^b\Big]\,. \tag{221}$$

The final Lagrangian is therefore

$$
\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{2}\partial_\mu r'\partial^\mu r' + \frac{1}{2}(r'+\rho)^2\left(n_a n_b + \frac{\sin^2\theta}{\theta^2}(\delta_{ab}-n_a n_b)\right)\partial_\mu\theta^a\partial^\mu\theta^b \\
& -\mu^2(r'^2+2\rho r')^2\,. \qquad (222)
\end{aligned}
$$

Comparing this with (203), we see there are some similarities. There are again, as expected, two massless fields - the θ^a are Goldstone bosons, while the third field again has mass $2\sqrt{2}\mu\rho$. Also, in both Lagrangians, there are no constant terms and no linear terms in the potential. It can be seen that this will be the case for any potential with minima $V=0$ at a finite value of r by expanding V in a Taylor series around this value of r[31, 66].

However, the fact that the θ^a form a realisation of O(3) on their own makes the r', θ^a coordinate system better for studying the low-energy limit: at energies much lower than the r' mass, (222) simply reduces to that for the O(3) non-linear sigma model, retaining its O(3) invariance.

Finally, we note that with the r' field included, it is still possible to couple the Lagrangian (222) to an auxilliary SO(2) gauge field by adding any bilinear function of $\mathbf{v}_\mu - eA^3_\mu(\mathrm{i}T_3)$ such as

$$
f(\theta, M)(\mathbf{v}_\mu - e\mathbf{A}_\mu)^a{}_b\theta^b(\mathbf{v}^\mu - e\mathbf{A}^\mu)_{ac}\theta^c \qquad (223)
$$

as discussed in Section 7.

12 Including gauge fields - the Higgs mechanism

We now wish to couple our triplet M^i to a set of Yang-Mills gauge fields transforming as the adjoint representation of SO(3). We do this by replacing $\partial_\mu M^i$ in (200) with covariant derivatives

$$
\mathrm{D}_\mu M^i = \partial_\mu M^i - eA^k_\mu(\mathrm{i}T_k)^i{}_j M^j\,. \qquad (224)
$$

To simplify the calculations, we follow Michel and Radicati[42] and write an element of the adjoint representation as an operator on vectors of the Lie algebra:

$$
(f_a)^i{}_j \equiv a^k\epsilon^i{}_{kj} = -a^k(\mathrm{i}T_k)^i{}_j \in \mathrm{so}(3) \qquad (225)
$$

thus using (207) the covariant derivative may be written

$$
\mathrm{D}_\mu M^i = (L\partial_\mu + (\partial_\mu\mathbf{L}) + ef_{A_\mu}L)^i{}_j m^j\,. \qquad (226)
$$

The length of this vector is invariant under the action of L^{-1}:

$$
(\mathrm{D}_\mu\mathbf{M}, \mathrm{D}^\mu\mathbf{M})\big|_M = (L^{-1}\mathrm{D}_\mu\mathbf{M}, L^{-1}\mathrm{D}^\mu\mathbf{M})\big|_m \qquad (227)
$$

so we define a new gauge field

$$
B^k_\mu(\mathrm{i}T_k) = L^{-1}A^k_\mu(\mathrm{i}T_k)L - \frac{1}{e}L^{-1}\partial_\mu\mathbf{L}\,. \qquad (228)
$$

Then

$$f_{B_\mu} = -\mathrm{i} B_\mu^k T_k = \frac{1}{e} L^{-1} \partial_\mu \mathbf{L} + L^{-1} f_{A_\mu} L \tag{229}$$

so we find

$$(\mathrm{D}_\mu \mathbf{M}, \mathrm{D}^\mu \mathbf{M}) = (\partial_\mu \mathbf{m} + e f_{B_\mu} \mathbf{m}, \partial^\mu \mathbf{m} + e f_{B^\mu} \mathbf{m})\,. \tag{230}$$

Calculating this inner product, remembering that $m^a = 0$ and $m^3 = r = r' + \rho$, we get

$$(\mathrm{D}_\mu \mathbf{M}, \mathrm{D}^\mu \mathbf{M}) = \partial_\mu r' \partial^\mu r' + e^2 B_\mu^a B_a^\mu (r' + \rho)^2\,. \tag{231}$$

We also expect there to be a kinetic term for the gauge fields. In the above notation, this may be written

$$\begin{aligned}(\mathbf{F}_{\mu\nu}, \mathbf{F}^{\mu\nu}) &= (\partial_\mu f_{A_\nu} - \partial_\nu f_{A_\mu} + e[f_{A_\mu}, f_{A_\nu}], \partial^\mu f_{A^\nu} - \partial^\nu f_{A^\mu} \\ &\quad + e[f_{A^\mu}, f_{A^\nu}]) \qquad (232)\\ &= \frac{1}{2}\,\mathbf{tr}[L^{-1}(\partial_\mu f_{A_\nu} - \partial_\nu f_{A_\mu} + e[f_{A_\mu}, f_{A_\nu}])L \\ &\quad \times L^{-1}(\partial^\mu f_{A^\nu} - \partial^\nu f_{A^\mu} + e[f_{A^\mu}, f_{A^\nu}])L] \qquad (233)\end{aligned}$$

Now from (229) we have

$$f_{A_\nu} = L f_{B_\nu} L^{-1} - \frac{1}{e}(\partial_\nu \mathbf{L}) L^{-1} \tag{234}$$

so

$$\begin{aligned}L^{-1}\partial_\mu f_{A_\nu} L &= (L^{-1}\partial_\mu \mathbf{L}) f_{B_\nu} + \partial_\mu f_{B_\nu} + f_{B_\nu} \partial_\mu (L^{-1}) L \\ &\quad - \frac{1}{e} L^{-1}(\partial_\mu \partial_\nu \mathbf{L}) - \frac{1}{e}(L^{-1}\partial_\nu \mathbf{L}) \partial_\mu (L^{-1}) L\,.\end{aligned} \tag{235}$$

It is easy to show that

$$\partial_\mu (L^{-1}) L = -L^{-1} \partial_\mu \mathbf{L} \tag{236}$$

so we find

$$\begin{aligned}L^{-1}(\partial_\mu f_{A_\nu} - \partial_\nu f_{A_\mu}) L &= \partial_\mu f_{B_\nu} - \partial_\nu f_{B_\mu} + [L^{-1}\partial_\mu \mathbf{L}, f_{B_\nu}] \\ &\quad - [L^{-1}\partial_\nu \mathbf{L}, f_{B_\mu}] + \frac{1}{e}[L^{-1}\partial_\nu \mathbf{L}, L^{-1}\partial_\mu \mathbf{L}].\end{aligned} \tag{237}$$

Similarly, we can use (234) to show that

$$\begin{aligned}e L^{-1}[f_{A_\mu}, f_{A_\nu}] L = e[f_{B_\mu}, f_{B_\nu}] - [L^{-1}\partial_\mu \mathbf{L}, f_{B_\nu}] + [L^{-1}\partial_\nu \mathbf{L}, f_{B_\mu}] \\ + \frac{1}{e}[L^{-1}\partial_\mu \mathbf{L}, L^{-1}\partial_\nu \mathbf{L}]\,.\end{aligned} \tag{238}$$

Substituting these results into (233) gives us

$$(\mathbf{F}_{\mu\nu}, \mathbf{F}^{\mu\nu}) = (\mathbf{B}_{\mu\nu}, \mathbf{B}^{\mu\nu}) \tag{239}$$

where $\mathbf{B}_{\mu\nu}$ is the field strength tensor of the $\mathbf{B}$ field.

The full broken Lagrangian is therefore

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu r' \partial^\mu r' - \frac{1}{4}(\mathbf{B}_{\mu\nu}, \mathbf{B}^{\mu\nu}) + \frac{1}{2}e^2\rho^2 B^a_\mu B^\mu_a + e^2 \rho r' B^a_\mu B^\mu_a \\ &\quad + \frac{1}{2}e^2 r'^2 B^a_\mu B^\mu_a - V(r' + \rho)\,. \end{aligned} \tag{240}$$

Note that r' interacts only with the B^a_μ fields and, as expected from the work of Higgs and Kibble, these gauge fields have received a mass through the Higgs mechanism, while B^3_μ remains massless. The Goldstone fields are no longer present in the Lagrangian, but appear in the relations between the high- and low-energy gauge fields, (228) and its inverse

$$A^k_\mu(\mathrm{i}T_k) = LB^k_\mu(\mathrm{i}T_k)L^{-1} + \frac{1}{e}(\partial_\mu \mathbf{L})L^{-1}\,. \tag{241}$$

This is obviously not ideal: we would like to express B^k_μ in terms of A^k_μ and M^i only, and express A^k_μ in terms of B^k_μ and r' only. However, even in this simple case, trying to eliminate the Goldstone fields from these expressions proves very messy and it is certainly not clear how to do this in a way which could be generalised to other symmetry groups. The solutions of this model are discussed briefly in Section (14.2).

13 Fermions and supersymmetry - review of methods

13.1 Minimal coupling to fermions - global O(3)

The model we presented in Section 11.2 can be extended by introducing addtional multiplets of global SO(3) - in particular, one may wish to add fermions. The simplest way to do this is to add to the Lagrangian (200) kinetic and mass terms for the new fields Ψ^i and terms describing the interaction with the M^i. Let us assume that these transform as a representation Γ of SO(3) [11]. Following the method of Salam and Strathdee[18] one then defines new field variables $\psi = \Gamma(L^{-1})\Psi$. If spontaneous symmetry breaking then occurs, these are then coupled to the Goldstone fields by 'covariant derivatives' involving $\mathbf{v}_\mu$[59]. Any self-coupling terms for the Ψ^i are invariant under the field redefinition so the self-couplings for the ψ^i are identical in form, e.g.

$$\Psi^i\Psi_i = \psi^i\psi_i \tag{242}$$

The original coupling terms between the Ψ^i and the M^i become couplings between the ψ^i and r'. The ψ multiplet breaks into multiplets of SO(2). They do not form a realisation of SO(3) on their own but only as part of a larger realisation including the Goldstone fields[5].

[11] The geometry of spin structures on coset spaces has been studied by Balachandran *et al*[67].

13.2 Fermionic coupling to the gauge theory

We can also extend the gauged model by introducing the additional fields Ψ. Again we add to the linear Lagrangian kinetic and self-interaction terms for these fields and terms for the interaction between them and the M^i. To couple them to the gauge fields we make the replacement

$$\partial_\mu \Psi \to \mathrm{D}_\mu \Psi \equiv \partial_\mu \Psi + q A^i_\mu \Gamma(T_i)\Psi \tag{243}$$

in their kinetic terms, where the charges q may be different for each multiplet. On making the field redefinitions $\psi = \Gamma(L^{-1})\Psi$ and $r' = m^3 - \rho$, we again find that the self-interactions for the Ψ become self-interactions for the ψ and interactions between the Ψ and the M^i become interactions between the ψ and r'. The couplings to the SO(3) gauge fields become couplings to the field B^3_μ[18]:

$$\mathrm{D}_\mu \Psi \equiv \partial_\mu \Psi + e A^i_\mu \Gamma(T_i)\Psi \to \mathrm{D}_\mu \psi \equiv \partial_\mu \psi + e B^3_\mu \Gamma(T_3)\psi \,. \tag{244}$$

13.3 Supersymmetry - global O(3)

In recent years, there has been a great deal of interest in supersymmetric sigma models and symmetry breaking in supersymmetric theories. In supersymmetric theories, there are generators which transform as Lorentz spinors and as a multiplet of an internal symmetry. These generate transformations between fermions and bosons in 'superspace'. The full algebra contains both commutation and anticommutation relations and involves the spinorial generators, the Poincaré (or conformal) generators and the internal symmetry generators. The multiplets ('superfields') contain both fermionic and bosonic components.

It is possible to extend our model to admit N=1 supersymmetry in four dimensions[68] or N=2 supersymmetry in two dimensions[69, 70]. In this section, we highlight some aspects of the four-dimensional supersymmetric model, particularly those that have rarely been emphasised in past treatments.

13.3.1 Superfields

In the four-dimensional model, the multiplet M^i is replaced as the basic multiplet of the linear theory with a multiplet of chiral superfields

$$\Pi^i = z^i(y) + \theta\lambda^i_\Pi(y) + \theta^2 Z^i_\Pi(y) \tag{245}$$

where

$$z^i = A^i + \mathrm{i}B^i \,, \qquad Z^i_\Pi = F^i_\Pi - G^i_\Pi \,, \tag{246}$$

$$y^\mu = x^\mu + \mathrm{i}\theta\sigma^\mu\bar{\theta} \,, \qquad \sigma^\mu = (-\mathbb{1}, \sigma^i) \tag{247}$$

and θ now represents the superspace parameter. A^i is a Lorentz scalar, B^i is a pseudoscalar, λ^i is a spinor and F^i and G^i are auxilliary fields.

Supersymmetric actions are composed of integrals of products of such superfields over both real space and superspace. By integrating over superspace and using the equations of motion to eliminate the auxilliary fields, one then obtains a more familiar form of the action. In our case, we would like this latter form of the action to contain a potential similar to (201); however, the supersymmetry requirements are too restrictive to obtain this from an action just involving Π^i. Barnes *et al*[71] get round this by introducing a 'spectator' superfield,

$$\Phi^i = \phi(y) + \theta\lambda_\Phi(y) + \theta^2 Z_\Phi(y) \,. \tag{248}$$

The action is then

$$I = \int \mathrm{d}^4x\, \mathrm{d}^4\theta(\bar{\Pi}^i\Pi_i + \bar{\Phi}\Phi) + \int \mathrm{d}^4x\, \mathrm{d}^2\theta W(\Pi^i, \Phi) + \int \mathrm{d}^4x\, \mathrm{d}^2\bar{\theta}\bar{W}(\bar{\Pi}^i, \bar{\Phi}) \tag{249}$$

with

$$W = k(\Pi^i\Pi_i - \rho^2)\Phi \tag{250}$$

where k is a numerical constant.

Carrying out the superspace integration of W and eliminating the auxilliary fields yields the potential

$$V = 4k^2\phi\bar{\phi}z^i\bar{z}_i + k^2(z^iz_i - \rho^2)(\bar{z}^i\bar{z}_i - \rho^2) \,. \tag{251}$$

One minimum point of this potential is

$$\begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} = \begin{pmatrix} \bar{z}^1 \\ \bar{z}^2 \\ \bar{z}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix} \,. \tag{252}$$

13.3.2 Symmetries of the scalar sector

Note that z^i is a *complex* multiplet. Its components are transformed into each other by the *complexification* of SO(3), denoted SO(3)^c. Bando *et al*[72] point out the particular importance of nilpotent matrices in the algebra of this group. We follow Higashijima and Nitta[73] and take the real generators of SO(3)^c to be

$$UT_iU^\dagger$$

where

$$U = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{\mathrm{i}}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} , \tag{253}$$

i.e.

$$T'_1 = \begin{pmatrix} 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ -\frac{\mathrm{i}}{\sqrt{2}} & 0 & -\frac{\mathrm{i}}{\sqrt{2}} \\ 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 \end{pmatrix} , \tag{254}$$

$$T'_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{255}$$

$$T_3' = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{256}$$

The compact subgroup generated by T_2' multiplies z^1 and z^3 by a constant phase factor, that is it mixes the scalars with their pseudoscalar partners. The noncompact subgroup generated by $\mathrm{i}T_2'$ multiplies z^1 by real scalar factor and divides z^3 by the same factor. Under the transformation 'generated' by the nilpotent combinations $T_3' + \mathrm{i}T_1'$ and $\mathrm{i}(T_3' + \mathrm{i}T_1')$, z^1 is invariant, while under those 'generated' by the conjugates of these, z^3 is invariant.

The point (252) on the complex vacuum manifold is only mapped to minima with non-zero values of z^1 and/or z^2 by transformations 'generated' by $T_3' - \mathrm{i}T_1'$ and $\mathrm{i}(T_3' - \mathrm{i}T_1')$. Of the other four transformations, one multiplies this minimum by a scale factor, one rotates it in the complex plane and two are true invariances of this minimum.

13.3.3 Other comments

It is worth noting that requiring an initial Lagrangian such as (249) to be manifestly supersymmetric greatly constrains its form (and hence that of the scalar potential V). If one is trying to find a model with particular features, this could be problematic, but if one is looking for uniqueness, this could be seen as an advantage.

Another noteworthy feature of manifestly supersymmetric Lagrangians is the lack of explicit spacetime derivatives: the derivatives of the individual components are instead implicit in the superspace integrals, which are very easy to handle. In this regard, it could be argued that the supersymmetric model is rather more elegant than its non-supersymmetric counterpart.

A supersymmetric version of Salam and Strathdee's method for arbitrary G and H cannot be constructed, as the low-energy field content is heavily dependent on the geometry of G/H. The simplest cases, such as that above, are where G/H is Kähler[74, 72, 75]. In such cases, the Goldstone bosons are not accompanied by 'quasi-Nambu-Goldstone' superpartners and the relation between an element of $G^c/\hat{H}$ (the coset space generated by $T_3' + \mathrm{i}T_1'$ and $\mathrm{i}(T_3' + \mathrm{i}T_1')$ in the above example) and the corresponding element of G/H is well known[75].

Many of these ideas have been thoroughly explored by Higashijima and Nitta[76, 77, 78]. For our example, we can get some idea of the low-energy limit from these papers and from Barnes[79] . As $\mathrm{SO}(3)/\mathrm{SO}(2) \approx \mathrm{SU}(2)/\mathrm{U}(1)$ is a Kähler manifold, the scalar part of the low-energy action may be written[74]

$$I_{\mathrm{scalar}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K \tag{257}$$

where, following Itoh *et al*[75], the Kähler potential K is a function of just one complex scalar γ. The fermionic part of the action seems to be more difficult to predict without explicit calculation, but we would expect γ to have a massless partner[80]. Once again, this is equivalent to a theory with an auxilliary gauge field[28, 81, 82, 83, 84, 85] (in this case a vector superfield).

13.4 Supersymmetry in the gauge theory

The supersymmetric Higgs mechanism was first considered by Fayet and Iliopoulos [86, 87]. The second of these papers contains a model in which supersymmetry is preserved but gauge invariance is broken by use of a 'spectator' superfield. We can achieve this simply by gauging the action (249) with a vector superfield multiplet V^i transforming as a triplet of SO(3)[87, 73]. However, Fayet later showed that with the internal symmetry gauged by a vector superfield, neither a spectator superfield nor a superpotential are required for spontaneous symmetry breaking. This is because, with an action

$$I = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\; \bar{\Pi}_i (\mathrm{e}^{2g(\mathbf{V})})^i{}_j \Pi^j + \int \mathrm{d}^4x\, \mathrm{d}^2\theta\; \frac{1}{32} W^\alpha W_\alpha \tag{258}$$

just composed of covariantised kinetic terms, the auxilliary fields of V^i modify the scalar potential giving it degenerate minima. These minima correspond to all possible values of z^i, spanning $\mathbb{C}^3$. However, if we consider the complexification of S^2 described above containing the point (252), every point in this space is related by an SU(2) gauge transformation[88, 13]. The value of ρ therefore distinguishes between gauge inequivalent vacua. The real and imaginery parts of z^3 can be used as coordinates at large ρ.

Furthermore, the action (258) has $N = 2$ supersymmetry, with Π^i and V^i together forming a single $N = 2$ superfield[89]. Upto a homomorphism, this is the system studied so successfully for varying values of ρ by Seiberg and Witten[13] by utilising duality.[12]

14 Quantisation and renormalisation - summary of known results

14.1 Global symmetry

Several methods of quantisation have been applied to the sigma model, including Dirac, multi-Hamilton-Jacobi and Batalin-Fradkin-Tyutin[91, 92]. In the low-energy limit the metric is non-polynomial and for such a theory in more than two spacetime dimensions, Feynman diagrams with higher numbers of loops are increasingly divergent. However, these correspond to higher momenta[93]. At low energies we only need to consider tree diagrams. For the bare sigma model, terms in the power series expansion of the metric can always be characterised by a 'coupling constant' a, if necessary by rescaling the fields[5]. (For example, for projective coordinates, an expansion in the bare coupling constant $a = 1/M^2$ is given in equation (48).) It can also be shown that for the Feynman diagrams, the power of a rises in proportion to the number of loops, so only the first few terms in the metric expansion are needed for tree diagrams[5, 19].

Quantisation and renormalisation usually imply the presence of extra terms in the Lagrangian. With one spacelike dimension, quantisation leads to the

[12] These aspects of Seiberg-Witten theory are described with great clarity by Bilal[90].

appearance of the soliton term, while with two, the Hopf term emerges[35, 34]. The model is then equivalent to one involving higher-spin fields[94]. In Minkowski space, renormalising to one loop requires adding counterterms[93]; in the form of the sigma model with an auxilliary gauge field these include a kinetic term for the gauge field so it becomes a real, dynamical field[30].

At higher energies, we would expect the linear SO(3) symmetry to reappear, and indeed, when the renormalised coupling constant exceeds a critical value, corresponding to a critical temperature, a third field component appears, forming a linear multiplet with the others[24, 25, 95]. Such a phase transition seems to hold to all orders in four dimensions[25, 96]. However, in two spacetime dimensions, the critical value reduces to zero, so quantum corrections ensure symmetry remains unbroken at all temperatures. Also, it should be noted that dynamical processes associated with coupling a non-linear sigma model to fermions may affect the symmetry breaking[97].

14.2 Gauged symmetry

It is well known that Yang-Mills theories are renormalisable and that this is not affected by spontaneous symmetry breaking[98]. Furthermore, SO(3) Yang-Mills theories are free of Adler-Bell-Jackiw anomalies[99].

In d dimensions, the classical solutions and, at the quantum level, the non-perturbative spectrum of this model, are determined by the homotopy group $\pi_{d-1}(S^2)$. For $d = 2$ this is trivial, but for $d = 3, 4$ it is $\mathbb{Z}$[100]. BPS states then appear - 'Hooft-Polyakov monopoles[101, 102, 103, 104] and dyons[105, 104]. Indeed, in three dimensions, the **B** fields see a background plasma of monopoles, which endow the B^3 field with a mass and linearly confine the other two. (There are other perspectives on this and Kogan and Kovner have looked at the phase transition from a number of viewpoints[106].)

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