Tangent space symmetries in general relativity and teleparallelism

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Abstract

This paper looks at connections on a pseudo-Riemannian manifold and the symmetries of its tangent spaces. In particular, it looks at a coset decomposition of the general linear group of Jacobian matrices, and the relationship between this, the Levi-Civita connection and the Weitzenböck connection. It then addresses the role of translations in general relativity and its teleparallel equivalent, which has been the subject of recent debate.

1 Introduction

General relativity demonstrates the importance of geometry to our understanding of fundamental physics. The basic variables in the gravitational sector of the theory are the independent components of the metric. In an N-dimensional spacetime, there are $\frac{N(N+1)}{2}$ of these. Another key quantity is the connection - general relativity uses the Levi-Civita connection, which is uniquely defined on a given spacetime for a given coordinate system. The theory has been extraordinarily successful in describing the action of gravity.

When the theory was developed, the only other fundamental interaction that was recognised by physics was electromagnetism. Einstein was keen to extend the theory to incorporate electromagnetism in a geometric way. He tried a number of different approaches to this[1].

One approach was "Fernparallelismus", often called "distant parallelism" or "teleparallelism"[2]. He noted that an N-bein field has N^2 independent components. The components of the metric can be written as functions of these, but then there are $\frac{N(N-1)}{2}$ degrees of freedom contained in the N-bein field which describe invariances of the metric[3]. His idea was that these additional degrees of freedom could be used in describing electromagnetism. In defining an N-bein field across the spacetime manifold, he needed to use a new type of connection, which he discovered had already been investigated by Cartan, Weitzenböck and others[2, 4].

While this approach was unsuccessful in its aim, research into teleparallelism and its application to gravity has continued. It is now known that a theory of gravity can be based on the principles of teleparallelism which reproduces the field equations of general relativity. This is known as the Teleparallel Equivalent of General Relativity (TEGR). TEGR is particularly useful for calculations concerning conservation of energy density, due to the way in which the energy-momentum density of the gravitation field may be separated out from inertial effects[5]. (This cannot be done in general relativity because these inertial effects depend on the choice of frame, so cannot be represented by a tensor.)

Much of the research presented here has its foundations in a TEGR paper by Pereira[6]. As well as presenting original research, this provided an excellent summary the subject to that point. Further detail on the theory developed by 2013 can be found in another review by Maluf[7].

More recently, the role of translation symmetries in TEGR has become the focus of vigorous debate. The idea of a theory of gravity as a gauge theory of translations dates back at least to Hayashi and Nakano[8], who constructed a derivative operator which is covariant under local translations. Pereira and Obukhov (among others) claim that TEGR constitutes a gauge theory of translations [10, 11], whereas this is disputed by Fontanini *et al*[12, 13]. This question is addressed directly in the current paper.

The research presented here focuses on the action of symmetry groups on the tangent space, induced by changes of coordinates. It assumes the usual four-dimensional spacetime of general relativity, although the extension to an N-dimensional spacetime is straightforward. The tetrad field components are the elements of a matrix transformation which maps a chosen frame basis into a coordinate basis, as described below. These matrices form a general linear group. Furthermore, the invariances of the metric described by Einstein form a pseudo-orthogonal group, which is a subgroup of the general linear group. The general linear group can be partitioned into cosets of the pseudo-orthogonal group and this leads to a natural decomposition of the change of frame.

This paper looks at the relationship between the Weitzenböck and Levi-Civita connections and these tangent space symmetries, utilising the decomposition of the general linear group. It starts by looking at the analysis of the tangent space at a single point in spacetime. This analysis is then extended to a curve, allowing one to define connections and covariant derivatives. It is then extended further to a four-dimensional chart on the spacetime. Choices of connection and their associated covariant derivative are then available and we look at the features of these and the relationships between them.

It also looks at translations as changes of coordinates and their induced action on the tangent space at a point. It is shown for the tangent space, these do not constitute a separate set of symmetries. Global translations do not affect basis vectors or the components of a vector. The action on the tangent space induced by local translations is simply that of the general linear group, albeit represented as a displacement of the basis rather than a contraction with the basis. This displacement provides a minimal coupling to the derivatives of the translation parameters, in line with the covariant derivative operator of Hayashi and Nakano[8]. However, an inhomogeneous displacement of the vector components cannot be induced by a coordinate transformation. Allowing for such a transformation would constitute extending general relativity. The relationship between local translations and the general linear group allows one to write, for example, the Weitzenböck connection in terms of the derivatives of the translation parameters.

We follow the notation of Pereira[6] in using a dot above a connection or covariant derivative to specify that it is a Weitzenböck connection or covariant derivative. Similarly, circles above connections or covariant derivatives denote they are Levi-Civita ones.

Where we need to specify which coordinate system a set of tensor components relates to, we will do so by putting it in brackets in a superscript or subscript. For example, the components of a vector $\mathbf{V}$ in a coordinate system u'^M will be written $V^M_{(u')}$.

Where we are evaluating a quantity at a given point, we shall state explicitly which point it is evaluated at, to avoid confusion (which can be considerable) between the value of the quantity and the functional form of that quantity.

In the early parts of the paper, we take the arguments step-by-step from first principles. It may seem unnecessarily slow and basic to some readers, but this is what has led to the insights in this paper into the current debate, and the author prefers to err on the side of caution to avoid further misunderstandings.

2 The tangent space at a point

General relativity treats spacetime as a curved pseudo-Riemannian manifold. On such a manifold, one needs to use curvilinear coordinates to parametrise a finite region of it. For a given region Ω of any given manifold $\mathcal{M}$, there are an infinite number of possible curvilinear coordinate systems that could be used. One therefore needs rules for changing from one coordinate system to another. In general, it is assumed that if one starts with a coordinate system u^I, then any second set of coordinates u'^I can be written as analytic functions of u^I:

$$u'^I = f^I + f^I{}_J u^J + f^I{}_{JK} u^J u^K + \dots \tag{1}$$

where the coefficients $f^I{}_{JK\dots}$ are real and symmetric on their lower indices and independent of the coordinates.

General relativity is constructed to be generally covariant, meaning that equations can be expressed in forms independent of the precise coordinate system being used. This is achievable despite the complexity of the relationship between u^I and u'^I, because general relativity is expressed in terms of tangent vectors, tensors and connections. The fact that the spacetime is pseudo-Riemannian means that it approximates to flat spacetime at each point. This allows one to define a tangent space at each point, the elements of which are vectors. By taking outer products of the tangent spaces and their duals, one can define tensors of higher rank. These have much simpler transformation laws than the underlying coordinates do. However, all transformations that are applied to tensors are induced by coordinate transformations. This turns out to be crucial to a full understanding of the relationship between translations and general linear transformations.

The vectors tangent to the curves of increasing $u^0, u^1,\ u^2, u^3$ at a point A form a basis for the tangent space $T_A\mathcal{M}$, denoted $\mathbf{e}_M|_A$ - the "coordinate basis" for u^M. The value of a vector field at A may then be written as a linear sum of this coordinate basis

$$\mathbf{V}|_A \in T_A\mathcal{M} = V^M_{(u)}|_A \mathbf{e}_M|_A \tag{2}$$

Indeed, it can be written as a linear sum of any set of four independent vectors in the space. In particular, it may be written as a linear sum of a second coordinate basis:

$$\mathbf{V}|_A = V^M_{(u)}|_A \mathbf{e}_M|_A = V^N_{(u')}|_A \mathbf{e'}_N|_A \tag{3}$$

We find the relations between the two bases by considering two neighbouring points, A and B. If they are separated by an infinitesimal interval, the displacement is a vector in $T_A\mathcal{M}$. This may be written in the two coordinate systems as

$$du^M|_A\, \mathbf{e}_M|_A = du'^N|_A\, \mathbf{e'}_N|_A \tag{4}$$

Now B has coordinates $u^I = du^I$ and $u'^I + du'^I$. From (1), these are related by

$$u'^I + du'^I = f^I + f^I{}_J(u^J + du^J) + f^I{}_{JK}(u^J + du^J)(u^K + du^K) + \dots \tag{5}$$

so

$$du'^I = f^I{}_J du^J + f^I{}_{JK} u^J du^K + f^I{}_{JK} du^J u^K + \dots \tag{6}$$

while we can find from first principles that

$$\left.\frac{\partial u'^I}{\partial u^J}\right|_A = f^I{}_J + f^I{}_{JK} u^K + f^I{}_{KJ} u^K + \dots \tag{7}$$

Comparing these last two equations, we find, unsurprisingly, that

$$du'^I = \left.\frac{\partial u'^I}{\partial u^J}\right|_A du^J \tag{8}$$

Substituting this into (4) gives us

$$du^M|_A\, \mathbf{e}_M|_A = du^M|_A \left.\frac{\partial u'^N}{\partial u^M}\right|_A \mathbf{e'}_N|_A \tag{9}$$

This same transformation law is valid for any vector:

$$V^M_{(u)}|_A \mathbf{e}_M|_A = V^M_{(u)}|_A \left.\frac{\partial u'^N}{\partial u^M}\right|_A \mathbf{e'}_N|_A \tag{10}$$

We can see this as a transformation of either the basis:

$$\mathbf{e}_M|_A = \left.\frac{\partial u'^N}{\partial u^M}\right|_A \mathbf{e'}_N|_A \tag{11}$$

or the components:

$$V^M_{(u')}|_A = V^M_{(u)}|_A \left.\frac{\partial u'^N}{\partial u^M}\right|_A \tag{12}$$

Thus while the coordinate transformation (1) and the expression for the Jacobian matrix are (possibly infinite) series of polynomial terms, the actual rules for transforming bases (11) and vectors (12) are simple homogeneous linear equations. This simplification is arguably the greatest advantage of working with Riemannian or pseudo-Riemannian manifolds.

This being a pseudo-Riemannian manifold, we can also define a symmetric inner product for each tangent space:

$$(\mathbf{V}, \mathbf{W})_A = (\mathbf{W}, \mathbf{V})_A \in \mathbb{R} \tag{13}$$

The image of this map on the coordinate basis is the metric at A:

$$g_{MN}|_A = (\mathbf{e}_M, \mathbf{e}_N)_A \tag{14}$$

and the inner product acts linearly over the tangent space. We can use this to find the transformation of the metric under a change of coordinates.

We can always define a set of coordinates x^I for which the basis is pseudo-orthonormal at our chosen point (with respect to the inner product). We will call this "frame basis" $\hat{\mathbf{n}}_I$:

$$(\hat{\mathbf{n}}_I, \hat{\mathbf{n}}_J)_A = \eta_{IJ} \tag{15}$$

The Jacobian matrices for transforming between bases at A are elements of a group J_A which is isomorphic to $GL(4,\mathbb{R})$. We will denote the transformation between the chosen frame basis and the chosen (unprimed) coordinate basis $j_0|_A$:

$$(j_0)_M{}^I|_A = \left.\frac{\partial x^I}{\partial u^M}\right|_A \in J_A : \hat{\mathbf{n}}_M \mapsto \mathbf{e}_M = (j_0)_M{}^I \hat{\mathbf{n}}_I \tag{16}$$

while j will be used for a generic change of basis - for example,

$$j \in J_A : \mathbf{e}_M \mapsto \mathbf{e}'_M = j_M{}^N \mathbf{e}_N \tag{17}$$

Note that in this formalism, V^M consequently transforms according to:

$$j : V^M_{(u)}|_A \mapsto V^M_{(u')}|_A = V^N_{(u)}|_A (j^{-1})_N{}^M \tag{18}$$

As mentioned in the introduction, j_0 can be decomposed using a pseudo-orthogonal subgroup. The Minkowski metric (15) is invariant under spacetime rotations (including boosts) and spacetime inversions (such as reflections) and combinations of these, which make up a group I_A isomorphic to $O(1,3)$. J_A can be partitioned into cosets of the form $\lambda_0 I_A$, so we can always write

$$j_0|_A = \Lambda_0|_A i_0|_A \tag{19}$$

where $i_0 \in I_A$. If we then define

$$\hat{\mathbf{k}}_K|_A = (i_0)_K{}^I|_A \; \hat{\mathbf{n}}_I|_A \tag{20}$$

we find that

$$(\hat{\mathbf{k}}_K, \hat{\mathbf{k}}_L)_A = \eta_{KL} \tag{21}$$

and

$$\mathbf{e}_M|_A = (\Lambda_0)_M{}^K|_A \, \hat{\mathbf{k}}_K \tag{22}$$

and

$$g_{MN}|_A = (\Lambda_0)_M{}^K|_A \, (\Lambda_0)_N{}^L|_A \, \eta_{KL} \tag{23}$$

3 Connections and covariant derivatives along a curve

Having examined the tangent space at a given point A, we now want to look at comparing the tangent spaces at different points. To do this, we need to use a connection.

General relativity uses a particular connection, the Levi-Civita connection, or Christoffel symbol. This has the advantages of being symmetric and being uniquely defined - on a given manifold in a given coordinate system, its components are single-valued at each point. However, when considering frame bases as we are here, it makes more sense to introduce the concepts by starting with connections on a curve, which can be generalised either to the Levi-Civita connection and its associated spin connection, or to those of teleparallelism.

Consider a curve $c(\lambda)$ through Ω parametrised by the single variable λ. We take λ to be invariant under changes of coordinate. Pick two points on it A and B. We define any map between the tangent spaces $T_A\mathcal{M}$ and $T_B\mathcal{M}$ which preserves linearity and the inner product as a "parallel map". There are an infinite number of these.

Now choose frame bases at both points, $\hat{\mathbf{n}}_I|_A$ and $\hat{\mathbf{n}}_I|_B$. Denote the parallel map $\tilde{}$ for which the image of $\hat{\mathbf{n}}_I|_A$ is $\hat{\mathbf{n}}_I|_B$:

$$\tilde{} : T_A\mathcal{M} \quad \rightarrow \quad T_B\mathcal{M} \tag{24}$$

$$\tilde{} : \hat{\mathbf{n}}_I|_A \quad \mapsto \quad \hat{\mathbf{n}}_I|_B \tag{25}$$

Then as $\tilde{}$ is a linear map,

$$\tilde{} : \mathbf{e}_M|_A \mapsto \tilde{\mathbf{e}}_M = (j_0|_A \, {j_0}^{-1}|_B)_M{}^N \mathbf{e}_N|_B \tag{26}$$

In the teleparallism formalism, this is valid regardless of how close or far apart A and B are. However, we are looking to define a connection. We therefore take A and B to be close to each other (the interval between these events is small). We then note that we can also define parallel maps to and from all the points on $c(\lambda)$ between these points - this set of parallel maps along this section of the curve constitutes a "parallelism". We choose this such that the transformation j_0 from the frame basis to the coordinate basis varies continuously with λ. (This means that not only must the coordinate basis and the frame basis be related by the same group J all along the curve, but j_0 must be in the same connected component of J at all points.) This allows us to carry out a Taylor expansion of ${j_0}^{-1}$ in λ, giving us

$$\tilde{\mathbf{e}}_M = \left(\mathbf{1} + \delta\lambda \left. \left(j_0 \frac{\partial {j_0}^{-1}}{\partial \lambda} \right)_M{}^N \right|_A \right) \mathbf{e}_N|_B + \mathcal{O}^2(\lambda) \tag{27}$$

From the linear nature of the parallel map, we then find the image of any vector $\mathbf{V}$:

$$\tilde{} : \mathbf{V}|_A \mapsto \tilde{\mathbf{V}} = V^N|_A \mathbf{e}_N|_B + \delta\lambda\, V^M|_A \left. \left(j_0 \frac{\partial {j_0}^{-1}}{\partial \lambda} \right)_M{}^N \right|_A \mathbf{e}_N|_B + \mathcal{O}^2(\lambda) \tag{28}$$

The quantity in brackets is our archetypal connection (up to a change in sign):

$$\left. \left(\Gamma_\lambda^{(u)} \right)_M{}^N \right|_A \equiv - \left. \left(j_0 \frac{\partial j_0^{-1}}{\partial \lambda} \right)_M{}^N \right|_A = \left. \left(\frac{\partial j_0}{\partial \lambda} j_0^{-1} \right)_M{}^N \right|_A \tag{29}$$

Under a change of curvilinear coordinates, from u^K to u'^K, we simply replace j_0 in these expressions by $j j_0$, where

$$j_M{}^N = \frac{\partial u^N}{\partial u'^M} \tag{30}$$

giving us

$$j : \left. \left(\Gamma_\lambda^{(u)} \right)_M{}^N \right|_A \mapsto \left. \left(\Gamma_\lambda^{(u')} \right)_M{}^N \right|_A = \left(j \Gamma_\lambda j^{-1} \right)_M{}^N \Big|_A - \left. \left(j \frac{\partial j^{-1}}{\partial \lambda} \right)_M{}^N \right|_A \tag{31}$$

One possible change of coordinates is to the set x^I mentioned above, with pseudo-orthonormal basis at A. Then $j = j_0^{-1}$, so

$$\left. \left(\Gamma_\lambda^{(x)} \right)_M{}^N \right|_A = 0 \tag{32}$$

If $c(\lambda)$ is a geodesic, then x can have pseudo-orthonormal basis, and $\Gamma_\lambda^{(x)} = 0$, along the entire curve.

We can also look at changing parallelism. Consider a new parallelism $\bar{}$, which again preserves orthonormality, so that

$$\bar{} : \hat{\mathbf{n}}_I\big|_A \mapsto \bar{\mathbf{n}}_I\big|_B = i_I{}^J\, \hat{\mathbf{n}}_J\big|_B \tag{33}$$

If i is constant along $c(\lambda)$, Γ_λ is unaffected. But if i varies with λ (we take it to be in the same connected component of I at every point),

$$\bar{} : \left(\Gamma_\lambda\right)_M{}^N\big|_A \mapsto \left(\Gamma'_\lambda\right)_M{}^N\big|_A - \left. \left(j_0 i \frac{\partial i^{-1}}{\partial \lambda} j_0^{-1} \right)_M{}^N \right|_A \tag{34}$$

We can use (28) to define a covariant derivative:

$$D_\lambda V^N = \partial_\lambda V^N + V^M \left(\Gamma_\lambda\right)_M{}^N \tag{35}$$

It is easy to show that this transforms covariantly:

$$j : D^{(u)}_\lambda V^M_{(u)} \mapsto D^{(u')}_\lambda V^M_{(u')} = D^{(u)}_\lambda V^N_{(u)} \left(j^{-1}\right)_N{}^M \tag{36}$$

In the x coordinates, this simply becomes

$$D^{(x)}_\lambda V^M_{(x)} = \partial_\lambda V^M_{(x)} \tag{37}$$

4 Connections and covariant derivatives across Ω

It is possible to extend the way we defined Γ above to the whole of Ω. Rather than just defining a parallelism - a set of parallel maps - along a curve, we define a parallelism across the whole of Ω.[1] This results in j_0 becoming a field over u^I. We can then define a connection field using the same approach as in (27), except we now Taylor expand in each of the curvilinear coordinates; this is known as the Weitzenböck connection:

$$\dot{\Gamma}_{LN}{}^M(u) \equiv -\left(j_0 \partial_L j_0{}^{-1}\right)_N{}^M \equiv \left(\partial_L(j_0) j_0{}^{-1}\right)_N{}^M \tag{38}$$

This is not the most general connection. Other rules for parallel transporting a vector exist, which do not take this form. More generally,

$$\tilde{}: \mathbf{V}|_A \mapsto \tilde{\mathbf{V}} = V^N|_A \mathbf{e}_N|_B - \delta u^L \; V^M|_A \Gamma_{LM}{}^N \mathbf{e}_N|_B + \mathcal{O}(\delta u)^2 \tag{39}$$

The transformation of $\Gamma_{LM}{}^N$ under a local change of basis is similar to the transformation for Γ_λ, except that we now need to act on the index L:

$$j(u) : \Gamma^{(u)}_{LM}{}^N \mapsto \Gamma^{(u')}_{LM}{}^N = j_L{}^K \left(j \Gamma^{(u)}_K j^{-1}\right)_M{}^N - j_L{}^K \left(j \partial_K j^{-1}\right)_M{}^N \tag{40}$$

where $(\Gamma_L)_M{}^N \equiv \Gamma_{LM}{}^N$.

Just as for Γ_λ, we can apply a transformation j_0^{-1} to reduce the Weitzenböck connection to zero - except that we can now do it over the whole of Ω. However, on a curved manifold, the frame bases defined by

$$\hat{\mathbf{n}}_I = \left(j_0^{-1}\right)_I{}^M \mathbf{e}_M \tag{41}$$

at each point *do not represent the basis for any coordinate system.*

It is worth noting what happens on a geodesic in more detail. If we consider a point particle moving along a geodesic, we can always base a set of coordinates x^I on its rest frame. The geodesic is parametrised by τ, the particle's proper time, which is proportional to x^0:

$$x^0 = \mathrm{c}\tau \tag{42}$$

[1] Note that Riemannian and pseudo-Riemannian manifolds are not in general parallelizable. This means that a single parallelism cannot be used for the entire manifold. (For example, it is well known that most n-spheres are not parallelisable.) However, in this paper we are just concerned with parallelisms over a chart.

These coordinates are "Riemann normal coordinates": they have pseudo-orthonormal basis along the entire geodesic, and indeed the first derivatives of the metric are zero. By comparison with (32), we therefore have

$$\left.\dot{\Gamma}^{(x)}_{0M}{}^{N}\right|_{c(\lambda)} = 0 \tag{43}$$

For any connection $\Gamma^{(u)}_{LM}{}^{N}$, we may define the covariant derivative of a vector, with components

$$D_L V^N = \partial_L V^N + V^M \Gamma_{LM}{}^N \tag{44}$$

The covariant derivative at a point A is an element of $T_A\mathcal{M} \otimes T^*_A\mathcal{M}$. Under a local change of basis, the inhomogeneous term in the transformation of Γ is cancelled by the inhomogeneous term in the transformation of $\partial_L V^M$. Consequently, $D_L V^M$ transforms covariantly:

$$j : D^{(u)}_L V^M_{(u)} \mapsto D^{(u')}_L V^M_{(u')} = j_L{}^K D^{(u)}_K V^N_{(u)} \left(j^{-1}\right)_N{}^M \tag{45}$$

This can be extended in the normal way to tensors of other ranks.

It is easy to show that any connection for which (39) preserves the inner product of vectors is metric compatible, that is

$$D_L g^{MN} = 0 \tag{46}$$

However, it is not necessarily symmetric. For example, the Weitzenböck connection is metric compatible, but has a torsion:

$$\dot{T}_{LM}{}^N = \dot{\Gamma}_{LM}{}^N - \dot{\Gamma}_{ML}{}^N \neq 0 \tag{47}$$

The only symmetric, metric-compatible connection is the Levi-Civita connection:

$$\mathring{\Gamma}_{LM}{}^N = \mathring{\Gamma}_{ML}{}^N = \frac{1}{2} g^{NK} \left(\partial_K g_{LM} - \partial_L g_{KM} - \partial_M g_{KL}\right) \tag{48}$$

As shown by Pereira[6] and others, any non-symmetric connection, including the Weitzenböck connection, can be written as the sum of its contorsion and the Levi-Civita connection:

$$\Gamma_{LM}{}^N = \mathring{\Gamma}_{LM}{}^N + K_{LM}{}^N \tag{49}$$

where

$$K_{LM}{}^N = \frac{1}{2}\left(T_M{}^N{}_L + T_L{}^N{}_M - T_{LM}{}^N\right) \tag{50}$$

Now for any geodesic $c(\lambda)$, in the Riemann normal coordinates x^I, the derivatives of the metric are zero, so

$$\left.\mathring{\Gamma}^{(x)}_{LM}{}^{N}\right|_{c(\lambda)} = 0 \tag{51}$$

However, away from the geodesic the Levi-Civita connection is non-zero on a curved manifold, even in this coordinate system. Note that incorporating (43), we have

$$\dot{\Gamma}^{(x)}_{0M}{}^{N}\Big|_{c(\lambda)} = \mathring{\Gamma}^{(x)}_{0M}{}^{N}\Big|_{c(\lambda)} = 0 \tag{52}$$

We conclude this section by noting some further properties of the Weitzenböck and Levi-Civita connections. The Weitzenböck connection has zero field strength[6, 9]:

$$\partial_L \dot{\Gamma}_{NK}{}^{M} - \partial_N \dot{\Gamma}_{LK}{}^{M} + \dot{\Gamma}_{NK}{}^{J} \dot{\Gamma}_{LJ}{}^{M} - \dot{\Gamma}_{LK}{}^{J} \dot{\Gamma}_{NJ}{}^{M} = 0 \tag{53}$$

and (as noted above), it can be reduced to zero across Ω by a local change of basis. The scalar curvature (the Ricci scalar) may be constructed from its torsion tensor[6, 7]. For a given coordinate system on a given manifold, this connection is not unique - its definition depends on the parallelism chosen.

The field strength of the Levi-Civita connection is the Riemann curvature tensor:

$$R^{M}{}_{KLN} = \partial_L \mathring{\Gamma}_{NK}{}^{M} - \partial_N \mathring{\Gamma}_{LK}{}^{M} + \mathring{\Gamma}_{NK}{}^{J} \mathring{\Gamma}_{LJ}{}^{M} - \mathring{\Gamma}_{LK}{}^{J} \mathring{\Gamma}_{NJ}{}^{M} \tag{54}$$

and the connection cannot be reduced to zero across Ω by a local change of basis, except on a flat spacetime. For a given coordinate system on a given manifold, it is unique. The Riemann tensor can also be viewed in terms of the action of the covariant derivatives on a vector field:

$$[D_K, D_J]\, W^I = R^{I}{}_{LKJ} W^L \tag{55}$$

Finally, each connection $\Gamma_{LM}{}^{N}$ has an associated Lorentz connection or spin connection. Pereira[6] defines a Lorentz connection as a one-form assuming values in the Lie algebra of the Lorentz group. In N dimensions, this will be in the Lie algebra $SO(1, N-1)$. This means that at least two of its indices must be frame indices. It therefore has two forms, one of which has all three indices as frame indices, while the other has two frame indices and one coordinate index. In the formalism of this paper, the Lorentz connection with three frame indices is considered to be the usual connection in the frame basis. The frame basis at a point A is the basis at that point for some set of Riemann normal coordinates x^I, so we can write this connection at this point as $\Gamma^{(x)}_{LK}{}^{M}\Big|_{A}$. The form with two frame indices and one coordinate index is considered to be in a mix of two different bases. We shall write this as follows:

$$\omega_M{}^{IJ} T_{IJ} \in SO(1, N-1) \tag{56}$$

where the first index is taken to be a coordinate index and the last two are frame indices.

If we choose a frame $\hat{\mathbf{n}}_M$ at A related to the coordinate basis by (16) where j_0 can be decomposed using (19), any connection in the coordinate basis can be related to a Lorentz connection as follows:

$$\Gamma^{(u)}_{LM}{}^{N} = (\Lambda_0)_L{}^{K} \eta_{KI} \omega_M{}^{IJ} (\Lambda_0^{-1})_J{}^{N} + (\Lambda_0)_L{}^{K} \partial_M (\Lambda_0^{-1})_K{}^{N} \tag{57}$$

This equation can be inverted to give:

$$\omega_M{}^{IJ} = \eta^{IK}[(\Lambda_0^{-1})_K{}^L \Gamma^{(u)}_{ML}{}^N (\Lambda_0)_N{}^J + (\Lambda_0^{-1})_K{}^L \partial_M (\Lambda_0)_L{}^J] \tag{58}$$

Of course, this is not unique: any local change of frame $i(u)$ (including $i_0(u)$) results in another Lorentz connection. $\omega_M{}^{IJ}$ transforms under a local change of frame according to:

$$i(u) : \omega_M{}^{IJ} \mapsto \omega'_M{}^{IJ} = \left(i\omega_M i^{-1}\right)^{IJ} - \left(i\partial_M i^{-1}\right)^{IJ} \tag{59}$$

where frame indices are raised and lowered using η^{IK} and η_{IK}. It transforms under a change of curvilinear coordinates according to:

$$j(u) : \omega_M{}^{IJ} \mapsto \omega'_M{}^{IJ} = j_M{}^N \omega_N{}^{IJ} \tag{60}$$

For the Weitzenböck connection, (57) amounts to a Cartan decomposition using (19):

$$\dot{\Gamma}^{(u)}_{LM}{}^N = (\Lambda_0)_L{}^K \eta_{KI} \dot{\omega}_M{}^{IJ} (\Lambda_0^{-1})_J{}^N + (\Lambda_0)_L{}^K \partial_M (\Lambda_0^{-1})_K{}^N \tag{61}$$

where

$$\eta_{KI} \dot{\omega}_M{}^{IJ} = (i_0 \partial_M i_0^{-1})_K{}^J \tag{62}$$

This means that the Weitzenböck spin connection can be reduced to zero everywhere by a local change of frame, whereas the Levi-Civita spin connection cannot[6].

5 Translations

Consider the subset of coordinate transformations (1) for which all coefficients after the first two terms are zero - that is, the (global) inhomogeneous linear transformations:

$$u'^I = f^I + f^I{}_J u^J \tag{63}$$

For these, the Jacobian matrix is

$$\frac{\partial u'^I}{\partial u^K} = f^I{}_K \tag{64}$$

From this, we can immediately see that such a transformation is isometric if and only if $f^I{}_K$ is pseudo-orthogonal. Such transformations, with the general form

$$u'^I = f^I + i^I{}_J u^J \tag{65}$$

comprise the Poincaré group. Amongst these are the global translations

$$u'^I = f^I + u^I \tag{66}$$

for which the Jacobian matrix is a Kronecker delta, meaning that bases and vector components are untransformed.

We now want to consider what happens when the translation parameters are made spacetime-dependent. First, we consider the transformation of the basis on the tangent space. This transforms by contraction with the inverse Jacobian matrix, which is

$$\frac{\partial u^I}{\partial u'^K} = \delta^I_K - \frac{\partial f^I}{\partial u'^K} \tag{67}$$

so

$$f(u^J) : \mathbf{e}_K|_A \mapsto \mathbf{e}'_K|_A = \mathbf{e}_K|_A - \frac{\partial f^I}{\partial u'^K}\mathbf{e}_I|_A \tag{68}$$

Such a transformation may take us from a frame basis to a curvilinear coordinate basis:

$$\frac{\partial x^I}{\partial u^K} = \delta^I_K - \frac{\partial f_0^I}{\partial u^K} \tag{69}$$

(where f_0^I represents a translation from a frame basis - it is not related to ${f^I}_J$) and

$$f_0^I(u^J) : \hat{\mathbf{n}}_K|_A \mapsto \mathbf{e}_K|_A = \hat{\mathbf{n}}_K|_A - (\partial_K f_0^I)\,\hat{\mathbf{n}}_I|_A \tag{70}$$

Thus as the translation parameters become spacetime-dependent, the coordinate bases start to vary from the frame bases. As the partial derivative operator transforms in the same way (indeed, it can be seen as a representation of the basis), the same is true for this operator:

$$f_0^I(u^J) : \partial_K|_A \mapsto \partial'_K|_A = \partial_K|_A - (\partial_K f_0^I)\,\partial_I|_A \tag{71}$$

Thus for a scalar field ϕ,

$$f_0^I(u^J) : \partial_K\phi|_A \mapsto \partial'_K\phi|_A = \partial_K\phi|_A - (\partial_K f_0^I)\,\partial_I\phi|_A \tag{72}$$

- that is, we have a minimal coupling to the 16 variables of $\partial_K f_0^I$.

What we are doing here is to view the action of the general linear group from a new perspective. We previously considered the action of j_0 by contraction on the frame basis, (16). We now consider the displacement of the basis under this action:

$$\delta\mathbf{e}_K|_A = \mathbf{e}_K|_A - \hat{\mathbf{n}}_K|_A = [{(j_0|_A)_K}^I - {\delta_K}^I]\,\hat{\mathbf{n}}_I|_A \tag{73}$$

or in terms of the translation parameters f_0^I:

$$\delta\mathbf{e}_K|_A = -\partial_K f_0^I \hat{\mathbf{n}}_I|_A \tag{74}$$

Thus we see that the local translations of the coordinates induce *the same transformations of the tangent space* as described by the action of the general linear group. They are related by

$${(j_0|_A)_K}^I - {\delta_K}^I = -\partial_K f_0^I|_A \tag{75}$$

or, more generally,

$${j_K}^I - {\delta_K}^I = -\partial_K f_0^I \tag{76}$$

and therefore contain the same 16 degrees of freedom.

Having looked at the action of local translations on the basis, we now want to turn to the action on the components of a vector field. Some authors have represented translations as a displacement of these components. This seems an appropriate point at which to provide clarification on this issue. In general relativity, under a coordinate transformation, $\mathbf{V}|_A \in T_A\mathcal{M}$ remains $\mathbf{V}|_A \in T_A\mathcal{M}$; however, as we have seen above, it can be decomposed in different bases on that tangent space. Let us say that in two such bases, it has components V^M and $V'^M = V^M + \delta V^M$. As $\mathbf{e}_M$ is a complete linear basis for the tangent space, any other basis vectors can be related to it by (17). Thus

$$(V^M + \delta V^M)|_A \, \mathbf{e}'_M|_A = V^L (j^{-1})_L{}^M \mathbf{e}'_M|_A \tag{77}$$

from which we find

$$\delta V^M|_A = V^L[(j^{-1})_L{}^M - \delta^M_L] \tag{78}$$

Thus in general relativity, any displacements of the vector components induced by coordinate transformations have the original vector components as a factor. Inhomogenous transformations of the vector components are not symmetries of the theory. It is possible to extend general relativity to incorporate these additional symmetries - this results in Metric Affine Gravity, in which curvature and torsion are considered as independent fields, related with different degrees of freedom of gravity[14].

The issue is then to find an expression for j^{-1} in terms of the translation parameters. We do this using the local version of (66):

$$(j^{-1})_I{}^K = \frac{\partial u'^I}{\partial u^K} = \delta^I_K + \frac{\partial f^I}{\partial u^K} \tag{79}$$

or

$$(j_0^{-1})_I{}^K = \frac{\partial u^I}{\partial x^K} = \delta^I_K + \frac{\partial f_0^I}{\partial x^K} \tag{80}$$

Note that the non-trivial term on the right-hand side here is not that we have in (71), as the derivative is with respect to x^I. We can sort this out with an iterative procedure:

$$\begin{aligned}
(j_0^{-1})_I{}^K = \frac{\partial u^I}{\partial x^K} &= \delta^I_K + \frac{\partial f_0^I}{\partial u^L}\frac{\partial u^L}{\partial x^K} &(81)\\
&= \delta^I_K + \frac{\partial f_0^I}{\partial u^L}\left(\delta^L_K + \frac{\partial f_0^L}{\partial u^M}\frac{\partial u^M}{\partial x^K}\right) &(82)\\
&= \delta^I_K + \partial_K f_0^I + (\partial_K f_0^L)(\partial_L f_0^I) + \ldots &(83)
\end{aligned}$$

and similarly

$$(j^{-1})_I{}^K = \delta^I_K + \partial_K f^I + (\partial_K f^L)(\partial_L f^I) + \ldots \tag{84}$$

The implications of the convergence criteria for these series remains an open question.

These expressions, together with (75) and (76), can then be substituted into any equations in the earlier sections which contain j or j_0, to get equations containing the translation parameters. For example, the Weitzenböck (teleparallelism) covariant derivative is

$$\dot{D}_L V^M = \partial_L V^M + V^N \dot{\Gamma}_{LN}{}^M \tag{85}$$

where

$$\dot{\Gamma}_{LN}{}^M = -\partial_N \partial_L f_0^I \left(\delta_I^M + \partial_I f^M + (\partial_I f^J)(\partial_J f^M) + \ldots\right) \tag{86}$$

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